The "Fundamental Law of Active Management" Is No Law of Anything[1]

By
Richard O. Michaud, David N. Esch, and Robert O. Michaud[2]

Draft: June 7, 2017

[1] An earlier version with the title "The Fundamental Law of Mismanagement" by Richard Michaud and Robert Michaud was published as a New Frontier Advisors' Newsletter, July 2005..

[2] Richard O. Michaud is CEO of New Frontier Advisors, LLC, Boston, MA 02110; David N. Esch is Managing Director of Research of New Frontier Advisors LLC.; Robert O. Michaud is CIO of New Frontier Advisors, LLC.

Abstract

Grinold's "Fundamental Law" is widely used to validate adding securities to an optimization universe, adding factors to forecast return, trading more frequently, and reducing constraints. We show with simple examples followed by rigorous simulation proofs that these proposals are not reliable for adding investment value. This is because the Grinold formula ignores real world investment issues of estimation error and necessary constraints for practice. These flawed principles have been unchallenged by academics and practitioners for nearly twenty-five years and may adversely impact many hundreds of billions of dollars or more in current fund management.

Benchmarks arise naturally in judging asset manager competence and for meeting investment goals. An active investment manager typically claims to provide enhanced return on average relative to a given benchmark or index for a given level of residual risk. The information ratio (IR) – estimated return relative to benchmark per unit of residual risk or tracking error – is a convenient and ubiquitous framework for measuring the value of active investment strategies.

The Grinold (1989) "Fundamental Law of Active Management" asserts that the maximum attainable IR is approximately the product of the Information Coefficient (IC) times the square root of the breadth (BR) of the strategy.[3] The IC represents the manager's estimated correlation of forecast with ex post residual return while the BR represents the number of independent bets or factors associated with the strategy. Grinold and Kahn (1995, 1999) assert that the "law" provides a simple framework for enhancing active investment strategies. While a manager may have a relatively small amount of information or IC for a given strategy, performance can be enhanced by increasing BR or the number of independent bets in the strategy. In particular they state: "The message is clear: It takes only a modest amount of skill to win as long as that skill is deployed frequently and across a large number of stocks."[4] Their recommendations include increasing trading frequency, size of the optimization universe, and factors to models for forecasting return. Assumptions include: independent sources of information and IC the same for each added bet or increase in BR.

Clarke, de Silva and Thorley (2002, 2006) (CST) generalize the Grinold formula by introducing the "transfer coefficient" (TC) to the Grinold formula. TC is a scaling factor that measures how information in individual securities is "transferred" into optimized portfolios. TC represents a measure of the reduction in investment value from optimization constraints. This widely influential article has been used to promote many variations of hedge fund, long-short, alternative, and unconstrained investment strategies.[5]

A significant literature exists on applying the Grinold law and variations for rationalizing various active equity management strategies. Extensions include: Buckle (2004), Qian and Hua (2004), Zhou (2008), Gorman et al (2010), Ding (2010), Huiz and Derwall (2011). Industry tutorials and perspectives include Kahn (1997), Kroll et al (2005), Utermann (2013), Darnell and Ferguson (2014), Menchero (2017). Teachings include the Chartered Financial Analyst (CFA) Institute Level 2, the Chartered Alternative Investment Analyst (CAIA) Level 1 and many conferences and academic courses in finance. Texts discussing the formula and applications include Focardi and Fabozzi (2004), Jacobs and Levy (2008), Diderich (2009), Anson et al (2012), Schulmerich et al (2015). A very substantial fraction of globally professionally managed funds are estimated to employ optimized portfolio design principles that are applications of Grinold's "Fundamental Law."

---

[3] The Grinold formula is analytically derived and based on an inequality unconstrained maximization of quadratic utility. It should not be confused with Markowitz (1952, 1959) which assumes linear (inequality and equality) constrained portfolios and requires quadratic programming techniques to compute the MV efficient frontier. In particular, the Markowitz efficient frontier is generally a concave curve in a total or residual return framework while in Grinold (see e.g., GK 1995, p. 94) it is a straight line emanating from a zero residual risk and return benchmark portfolio. The Grinold derivation also assumes IC to be small, in the order of 0.1.

[4] GK (1995, Ch. 6, p. 130), also GK (1999, Ch. 6, p. 162).

[5] One example is Kroll et al (2005). Michaud (1993) was the first to note possible limitations of the long-short active equity optimization framework.

We show with simple examples followed by rigorous simulation proofs that these proposals for optimized portfolio design are invalid and often self-defeating. This is because estimation error and investment realistic constraints are ignored in the Grinold formula. The proposals are not reliable routes for adding investment value. These flawed principles have been unchallenged by academics and practitioners for nearly twenty-five years and possibly adversely impact many hundreds of billions of dollars or more of managed assets in current investment practice.

The outline of the paper is as follows. Section 1 presents the Grinold formula, the GK and CST prescriptions for active management with reference to the GK casino management rationale. Section 2 discusses the limitations of the GK and CST prescriptions from an intuitive investment perspective. Section 3 provides a discussion of properties of index-relative mean-variance (MV) optimization and previous simulation studies relevant to our results. Section 4 presents our Monte Carlo simulation studies that confirm that the principles associated with the fundamental law are invalid and likely self-defeating. Section 5 provides a summary and conclusions.

1.0 Grinold's Fundamental Law of Active Management

The Grinold (1989) formula is an approximate decomposition of the information ratio (IR) generally associated with active investment management. Grinold shows that the MV optimization of an inequality unconstrained residual return investment strategy is approximately proportional to the product of the square root of the breadth (BR) and the information correlation (IC).[6] Mathematically,

$$IR \cong IC * \sqrt{BR}$$

where IR = information ratio = (alpha) / (residual or active risk)
IC = information correlation (ex ante, ex post return correlation)
BR = breadth or number of independent sources of information.

The formula teaches that successful active management depends on both the information level of the forecasts and the breadth associated with the optimization strategy. However, Grinold and Kahn (GK) (1995, 1999) and Clarke, deSliva, and Thorley (CST) (2002, 2006) go further. They apply the Grinold formula to assert that only a modest amount of information (IC) is necessary to win the investment game simply by sufficiently increasing the number of assets in the optimization universe, the number of factors in a multiple valuation framework, more frequent trading and reducing optimization constraints.

GK use a casino roulette game to rationalize applications of the Grinold formula to asset management in practice.[7] The probability or IC of a winning play (for the casino) of the roulette game is small but more plays (breadth) lead to the likelihood of more wealth. However, there are important differences between the play of a roulette game in a casino and the play of an investment game in practice. In the casino context, the IC is stable, known, positive, and constant across plays. In an investment game the IC is unstable, has estimation error, and may be insignificantly or even negatively related to return. Plays of the investment game may not be

[6] The detailed derivation is given in GK, Ch. 6, and Technical Appendix.
[7] The casino roulette game framework is very consistent with the assumptions used in the Grinold derivation in GK (1995, 1999, Ch. 6. App.)

independent and increasing the number may be undesirable. While interesting the casino game rationale for rationalizing applications of the Grinold formula to investment practice is invalid.

There are two fundamental reasons for limitations of the principles associated with the Grinold law for practical asset management: 1) the formula ignores the impact of estimation error in investment information on out-of-sample optimized investment performance; 2) the formula assumes a quadratic utility unconstrained optimization framework that ignores the necessity of including investment realistic constraints required for defining portfolio optimality in practice.

2.0 Discussion of GK and CST Prescriptions

GK and CST propose four principles of optimized portfolio design for enhanced investment value in an index-relative MV optimization framework. We discuss the limitations of each prescription in turn from an intuitive point of view.

2.1 Large Optimization Universe Fallacy

GK argue that investment value increases with the size of the optimization universe conditional that the IC is roughly equal for all securities in the optimization universe. How realistic is this assumption?

For a small universe of securities the assumption of uniform average IC may be tenable. Small universes may be fairly homogeneous in character. However, for a large and expanding optimization universe, it seems untenable to assume uniform average IC across all subsets. Any manager will naturally use the securities with the best information first. While, theoretically, adding more assets may add marginally to breadth, all other things the same, it is also likely to result in less predictable securities and reduce the overall average IC level of the universe. A lower average IC may cancel any gains made from increasing breadth.

The issue can be framed in a more common practical setting. Consider an analyst suddenly asked to cover twice as many stocks. Given limitations of time and resources, it is highly unlikely that the analyst's average IC is the same for the expanded set of stocks. Issues of resources and time rationalize why analysts tend to specialize in areas of the market or managers in investment strategies that limit the number of securities that they cover. In practice many traditional managers limit the number of securities they include in their active portfolio to not much more than twenty or fifty. Except for relatively small asset universes, the average IC and overall level of IR may often be a decreasing function of the number of stocks in the optimization universe, all other things the same. Grinold and Kahn seem to be aware of these limitations, for example as suggested by their statement "The fundamental law says that more breadth is better, provided the skill can be maintained." Nevertheless, average IC and optimization universe size are inevitably related especially for large universes of assets.

2.2 Multiple Factor Model Fallacy

Large stock universe optimizations typically use indices such as the S&P500, Russell 1000 or even a global stock index as benchmarks. In this case individual analysis of each stock is generally infeasible and analysts typically rely on factor valuation frameworks for forecasting alpha. For example, stock rankings or valuations may be based in part on an earnings yield factor.[8] As GK note, if earnings yield is the only factor for ranking stocks, there is only one independent source of information and the breadth equals one.

In the Grinold formula, the IR increases with the number of independent positive significant factors in the multiple valuation forecast model. However, in practice, asset valuation factors are often highly correlated and may often be statistically insignificant providing dubious out-of-sample forecast value.[9] Finding factors that are reasonably uncorrelated and significantly positive relative to ex post return is no simple task.

Factors are often chosen from a small number of categories considered to be relatively uncorrelated and positively related to return such as value, momentum, quality, dividends, and discounted cash flow.[10] In experience, breadth of multiple valuation models is typically very limited and unlikely to be very much greater than five independent of the size of the optimization universe.[11,12] As in adding stocks to an optimization universe, adding factors at some point is likely to include increasingly unreliable factors that are likely to reduce, not increase, the average IC of an investment strategy.

Michaud (1990) provides a simple illustration of adding factors to a multiple valuation model. While adding investment significant factors related to return can be additive to IC, it can also be detrimental in practice. There is no free lunch. Adding factors can as easily reduce as well as enhance forecast value, and the number of factors that can be added while maintaining a desirable total IC is severely limited in practice.

2.3 Invest Often Fallacy

GK recommend increasing trading period frequency or "plays" of the investment game to increase the BR, and thus the IR of a MV optimized portfolio. The Grinold formula assumes trading decision period independence and constant IC level. However, almost all investment strategies have natural limits on trading frequency.[13] For example, an asset manager trading on book or earnings to price will have significant limitations increasing trading frequency smaller than a month or quarter. Reducing the trading period below some limit will generally reduce effectiveness while increasing trading costs.

---

[8] Some standard methods for converting rankings to a ratio scale to input to a portfolio optimizer include Farrell (1983) and references. Michaud (1998, Ch. 12) notes some common scaling errors.

[9] There is a limit to the number of independent investment significant factors even in many commercial risk models, often far less than ten.

[10] Standard methods such as principal component analysis for finding orthogonal risk factors are seldom also reliably related to return.

[11] See e.g., Michaud (1999).

[12] While principal component or factor analysis procedures for identifying orthogonal factors in a data set may be used, most studies find no more than five to ten investment significant identifiable factors that are also useful for investment practice.

[13] Special cases may include proprietary trading desk strategies where the information level is maintained at a reasonable level and trading costs are nearly non-existent. Other cases, such as high frequency and algorithmic trading are arguably not investment strategies but very low level IC trading pattern recognition relative to highly sophisticated automated liquidity exchange intermediation.

Fundamentally, trading frequency is limited by constraints on the investment process relative to investment style.[14] Deep value managers may often be reluctant to trade much more than once a year while growth stock managers may want to trade multiple times in a given year. Increased trading, to be effective, requires increasing the independence of the trading decision while maintaining the same level of skill. This will generally require increased resources, if feasible, all other things the same. The normal trading decision period should be sufficiently frequent, but not more so, in order to extract relatively independent reliable information for a given investment strategy and market conditions.

It is worth noting that the notion of normal trading period for an investment strategy does not imply strict calendar trading. Portfolio drift and market volatility relative to new optimal may require trading earlier or later than an investment strategy "normal" period. In addition a manager may need to consider trading whenever new information is available or client objectives have changed. Portfolio monitoring relative to a normal trading period including estimation error is further discussed in Michaud et al (2012).

2.4 Remove Constraints Fallacy

Markowitz's (1952, 1959) MV optimization can accommodate linear equality and inequality constraints. In actual investment practice, MV optimized portfolios typically include many linear constraints. This is because unconstrained MV optimized portfolios are often investment unintuitive and impractical. Constraints are often imposed to manage instability, ambiguity, poor diversification characteristics, and limit poor out-of-sample performance.[15] However, constraints added solely for marketing or cosmetic purposes may result in little, if any, investment value and may obstruct the deployment of useful information in risk-return estimates.

In general, inequality constraints are necessary in practice. Inequality constraints reflect the financial fact that even the largest financial institutions have economic shorting and leveraging limitations. Recently, Markowitz (2005) demonstrates the importance of practical linear inequality constraints in defining portfolio optimality for theoretical finance and the validity of many tools of practical investment management. Long-only constraints limit liability risk, a largely unmeasured factor in most portfolio risk models and often an institutional requirement. Regulatory considerations may often mandate the use of no-shorting inequality constraints. Performance benchmarks may often mandate index related sets of constraints for controlling and monitoring investment objectives. Moreover, inequality constraints limit the often negative impact of estimation error in out-of-sample performance (Frost and Savarino 1988).

3.0 Properties of index-relative MV optimization

The Grinold formula is a statement of index-relative MV optimization. Index-relative MV optimization is total return MV optimization with an index weight sum to zero constraint. The index has no risk and no return by definition. In the unconstrained case the IR efficient frontier is a straight line starting from the origin with slope equal to IR. In the sign constrained case the IR efficient frontier is a straight line rising from the origin with slope equal to max IR until the first pivot portfolio on the total return MV efficient frontier.

---

[14] Trading costs and market volatility are additional considerations.

[15] Michaud (1989).

Roll (1992) provides the classic critique of the IR active return MV optimization framework. He shows that IR optimized portfolios are dominated by portfolios in MV total return space unless the index is assumed total return MV efficient.[16] Merton (1987) provides a rational market framework for index-relative MV optimization. Under relatively straightforward conditions consistent with many active asset management strategies common benchmarks may often be considered total risk and return MV efficient. Presuming economic rational agents, we assume the index chosen is total risk and return MV efficient. This assumption is a best case scenario for the investment value of an active investment strategy.

Assuming Merton MV efficiency, all index-relative MV optimized portfolios are MV total return efficient. There are an infinite number of possible IR efficient portfolios to represent the investment value of an index-relative optimized investment strategy. This is because max IR efficiency is ambiguous. This is consistent with index-relative optimization practice where the IR maximized portfolio is defined relative to a specified level of tracking error to the benchmark. In order to compare the out-of-sample characteristics of an optimized investment strategy it is convenient to focus on one relevant optimized portfolio. Note that the MSR portfolio is IR efficient on both the unconstrained and sign constrained efficient frontier under our rationality assumptions. It is also a convenient single portfolio to represent portfolio optimality relative to the MV inputs. In the following the in-sample optimal MSR portfolio is used to represent the out-of-sample investment value of max IR investment strategies for both unconstrained and sign constrained investment strategies.

3.1 Testing GK and CST proposals

Investment managers often use a back test to demonstrate the likely value of a proposed investment strategy. In this procedure a factor or strategy is evaluated on how it performed for some historical data over some time period. While the benefit of a back test may be practicality, no reliable prospective information is possible by definition. Back tests are notorious for misleading investors, resulting in loss of wealth, careers, and dissolution of firms. Investors should be keenly aware of the serious limitations of any back test as evidence of the reliability of any factor relationship or investment strategy.[17]

A simulation study is a more reliable alternative framework for testing the value of optimized investment strategies. Such a procedure evaluates the likely out-of-sample performance of an in-sample optimized portfolio under many realistic investment scenarios.

In the following sections we explain the summary statistics used to evaluate the out-of-sample performance of investments following the prescriptions of fundamental law, describe the simulation test framework in greater detail, and discuss the results of relevant simulation experiments consistent with our results.

3.2 Portfolio simulation study framework

In a typical simulation study, a referee is assumed to know the true means, standard deviations, and correlations for a set of assets and consequently the true MSR for an optimized portfolio of those assets. The players do not know the referee's true MV parameters. The players receive simulated

[16] It is worth noting that the Roll (1992) results assume unconstrained MV optimization.
[17] Even long-term academic studies remain susceptible to the unreliability of results of any back test.

returns based on the referee's data, so they can only observe the truth obscured by estimation error, as is true for all real-world investment managers. The players then compute optimal weights for their inputs and send them back to the referee to score. The referee determines the estimated MSR for that simulation. The procedure is repeated many times for a range of referee simulated scenarios, and averages of MSRs computed for each player. In this way the out-of-sample performance of each strategy can be compared, and the better overall strategy determined.

3.3 MV optimization simulation studies

Jobson and Korkie (1981) provide the classic study of the effect of estimation error on the out-of-sample investment value of inequality unconstrained MV optimized portfolios.[18] In their study the referee's truth is based on historical MV inputs for twenty stocks. They compute simulated MV inputs reflecting five years of monthly return data. They find that the average of the true SRs, as measured by the referee, of simulated MSR optimal portfolios, was twenty-five percent of the MSR of the referee's optimal portfolio. In addition they show that equal weighting substantially outperforms the optimized portfolios.[19] They conclude that unconstrained MV optimization is not recommendable for practice.

Frost and Savarino (1988) perform a similar simulation study that compares inequality unconstrained to long-only MV optimized portfolios. They find that long-only MV optimized portfolios dominate inequality unconstrained. This is because sign constraints often limit the impact of estimation error on out-of-sample investment performance. Note that this unreferenced result contradicts CST. Economically realistic constraints often act like Bayesian priors focused on portfolio structure enforcing rules representing legitimate information not contained in the optimization inputs. Such restriction can mitigate estimation error in risk-return estimates implicitly by forcing the solutions towards more likely optimal portfolios.

Michaud and Michaud (1998, 2008a, 2008b) include a number of simulation tests to show that Michaud MV optimization has superior average out-of-sample performance relative to classic MV. In a related simulation study, Markowitz and Usmen (2003) showed that Michaud optimization had superior average out-of-sample performance relative to classic MV optimization even when the MV player had superior information.[20]

4.0 GK and CST Simulation Proofs

In this section we develop simulation studies to address the limitations we have discussed informally in prior sections of the GK and CST prescriptions for practice.

4.1 Simulating Breadth while Maintaining Information Levels

GK describe the benefits associated with their formula as consistent with those associated with multiple spins of a roulette wheel for a casino. The more spins the more on average the house wins even for a small amount of information. In the simulation we propose, the important deviation from the GK roulette wheel framework is that the probability of a win for the house is not stable

---

[18] Note that the JK study applies equivalently to inequality unconstrained quadratic utility portfolio optimization, a framework widely used in financial theory and for the development of many investment strategies.

[19] An equal weighted portfolio is a simple way to compare the optimality of unconstrained optimized portfolios.

[20] Harvey et al (2008) dispute the results in Markowitz and Usmen (2003). However Michaud and Michaud (2008c) note critical limitations in the Harvey et al (2008) study and as Harvey et al (2008b) acknowledge.

and known but includes estimation error for all the unknown parameters of the return distribution. Although in practice adding assets is not identical to adding breadth as specified by GK and CST, we construct our simulation experiment so that we are adding one unit of breadth for each asset added to the case. This is accomplished by adding each new asset independently to the system. For each new asset an entry is added to the referee's expected return vector, and a new row and column are added to the referee's covariance matrix. The simulated mean/variance estimates to be used as optimization inputs also gain estimates corresponding the referee's new information, without affecting the entries for the assets already in place for the mean or variance estimates, and only slightly changing (improving) the off-diagonal elements of the simulated covariance estimate through the newly available information about the single-factor component of the Ledoit and Wolf estimate. Further details of the covariance estimation can be found in Ledoit and Wolf (2004a), but the new asset definitely constitutes a unit of breadth because its information is being supplied independently and separately from the rest of the assets in the case. Thus any negative result we observe in our simulations will a fortiori apply to actual investment in practice. In our simulation framework diminishing added performance relative to the Fundamental Law's square root prediction cannot be explained as failure to add breadth when increasing universe size.

4.2 Simulation Methodology

We begin our task with defining a large sample of historical market data[21] which will be the basis for our simulations. The particular dataset is, in an important sense, wholly immaterial to our argument. All that is essential is that the master data set represents a realistic vector of expected returns and a full-rank covariance matrix associated with the largest sample size of the experiment. Except in one case noted further below, full rank covariance in all our simulations is assured with the use of the Ledoit and Wolf (2004a) covariance estimator.

In a simulation experiment, the mean and covariance of the data are described as the referee's truth. We Monte Carlo simulate returns for the largest set of assets assuming a multivariate normal distribution given the referee's mean and covariance matrix. From these returns, we calculate a sample mean and a Ledoit and Wolf (2004a) covariance estimate. By varying the number of periods of simulated returns, we can change the average information level of the estimated means and variances. We calculate the realized average IC of each set of inputs by averaging the sample correlations of the simulated mean (expected return) estimates with the referee's mean vectors. It may be surprising to learn that for our dataset, a 0.10 average IC is attained with only four periods of simulated observed returns. We present cases representing ICs of approximately 0.10, 0.20, and 0.30, corresponding to 4, 13, and 30 simulation periods of returns that were used to estimate the inputs. Of course, each case has a different measured IC, and because of the Monte Carlo nature of our experiment, the average realized ICs for each sample size are not exactly equal to their target values. Because of this we present the average realized IC for each target IC and portfolio size in a table below the figure showing simulation results. The observation sizes for each target IC were determined by calibrating the largest portfolio size (500) for the experiment. Because of the positively skewed distribution of the sample standard deviation in the denominator of the sample

[21] We use a recent history of US market data (1994-2013) of publically available data to create our master asset list and corresponding mean and variance parameters. We selected all the assets from the largest 1000 in market capitalization with contiguous data from the period, excluding returns greater than 50% or less than -50%. We were able to find 544 stocks that met our criteria. Parallel experiments with shorter histories were also run to investigate if selection bias affects results, with no positive findings, so we present the twenty year history here. Readers wishing to replicate our experiment can access our data at www.newfrontieradvisors.com/research/data.

correlation formula, the averages tend to be slightly lower for smaller sample sizes, although the realized ICs are still fairly close to their targets. Of course different datasets would probably require different numbers of return periods to attain similar average ICs.

The Ledoit and Wolf estimator guarantees no degradation of performance as universe size increases. This guarantees a full-rank covariance input even if the number of simulated returns is less than universe size. Managers may often use a factor model such as a commercial risk model or Fama-French (1992) framework to create a covariance matrix. Choosing an appropriate factor model and simulating factor observations are hotly debated topics, varying widely from academic studies to actual practice. We do not wish to enter that debate in this article. More can be found in Fan, Fan and Lv (2008). To allay any possible concern about rank deficiency affecting our results, we have performed an additional experiment with the referee's truth, which is an error-free covariance matrix in our simulation framework. This final experiment eliminates covariance estimation as a plausible explanation for performance relative to the fundamental law.[22] Results for this experiment appear in the appendix.

Given mean and variance inputs we create MV optimized portfolios via three methods: unconstrained maximum Sharpe ratio, maximum Sharpe ratio with positivity constraints, and equal weighting. Other portfolio construction methods are possible but not part of the scope of our study.[23] Out-of-sample Sharpe ratios are then calculated for each method using the referee's parameter estimates. These are different from the in-sample Sharpe ratios and unavailable except in the context of a simulation study. Generally the in-sample estimates, created from the manager's inputs, are far too optimistic, and are not displayed because they would dominate our exhibits.

Our displays cover two ranges in practice: asset allocation and equity portfolio optimization strategies. Asset allocation strategies typically include five to thirty securities and rarely more than fifty. On the other hand equity portfolio optimization strategies may include hundreds or even thousands of assets in the optimization universe. Active equity asset managers often claim an IC level of approximately 0.10. This is an important assumption in the derivation of the Grinold formula. However, it is also interesting to display cases with greater information, to show the impact of increased information on the relative performance of our optimization procedures.

4.3 Simulation Results

Figure 1 consists of three panels of simulation results for 0.1, 0.2, and 0.3 IC levels, for sizes of optimization universes ranging from 5 to 500 assets. Separate optimizations were performed within the 5-50 asset range in steps of 5 assets, and in the 50-500 range in steps of 50 assets. Each value presented on the graphs is averaged from 16,000 optimizations simulated from the selected universe. The three graphed series in each panel show progressions of average Sharpe ratios resulting from three different optimization methods. The "unconstrained" series displays the out-of-sample averages of the simulated unconstrained MSR portfolios, the "equal weight" series displays the average Sharpe ratios of equal weighted portfolios, and the "constrained" series reflects the average Sharpe ratios of out-of-sample simulated long-only MSR portfolios.

---

[22] We have included these experiments in the appendix. The results are not meaningfully different.

[23] One obvious case is to compute Michaud (1998) optimized portfolios.

The results of our experiments from Figure 1 demonstrate a definite failure of the GK and CST specifications of the fundamental law of management. Specifically, the improvement is far less than the predicted square root relationship with regard to breadth and as well as the predicted linear relationship for IC. Increasing the IC from 0.10 to 0.30 produces improvement in Sharpe ratio performance out-of-sample, but far less than the improvement predicted by the law. For smaller portfolio sizes consistent with asset allocation, at all information levels, the unconstrained portfolios dramatically underperform both sign constrained and equal weighting.[24] While adding assets increases the Sharpe ratios of unconstrained portfolios out-of-sample, the gain is minimal and, we will argue below, unrealistic. How positivity constraints help the optimization process depends on the quality of information and universe size but our results contradict the CST view that eliminating constraints must add investment value. The naïve analyst may think *a priori* that performance will increase because of increased IR forecasts calculated with estimates used in the optimization, but such in-sample calculation amounts to assuming perfect information and estimation ability, clearly unrealistic for investors of any skill level. Our results vividly demonstrate the hazards of ignoring estimation error when optimizing.

For larger portfolio sizes, the optimized cases often outperform the equal weighted case, with better performance for greater information levels and for positivity constraints. A second important difference from the smaller cases is that the benefit of positivity constraints depends crucially on the level of presumed forecast information. For a typical level of IC = 0.10, sign constrained large universe optimization provides enhanced performance relative to unconstrained for much of the size spectrum. For IC 0.30, the out-of-sample unconstrained performance nearly attains the level of the constrained cases for the largest sample size of 500 assets. Greater levels of IC would likely exhibit a crossover for which unconstrained optimization under estimation error outperformed constrained, but these experiments would clearly assume unrealistically clairvoyant forecasts.

4.4 Simulation Discussion

It should be stressed at this point that these experiments portray idealized versions of portfolio management, deliberately ignoring important performance-impacting inevitabilities of practice. Uniformly high-quality information is free in this experiment, where in practice coverage of additional securities will always incur costs and is likely to degrade the overall quality (IC) of the system. Real markets are unlikely to produce many independent returns coming from identical distributions, since markets are constantly shifting. The quality of information from manager estimates especially for increasingly large universes is likely to be significantly lower in practice than in our simulations. Moreover, trading costs and other frictions associated with the practice of portfolio management are completely neglected in this analysis.

Our deliberate optimism in setting up the simulation has important implications. The unconstrained cases would likely exhibit far poorer performance in practice. Since almost all of the assets in the simulation universe are likely to have some investment value, the investor is little harmed by putting portfolio weight on the “wrong” assets. In the real world, constraints may often limit the harm caused by misinformation. This effect was clearly demonstrated and measured in Jobson and Korkie (1981). In a truly chaotic world with a lot of estimation error and bias, the equal weighted portfolio, which uses no information to distinguish among assets, can be hard to beat.

[24] Note that our small universe size results confirm those of Jobson and Korkie (1981) and Frost and Savarino (1988).

Note that our simulations maintain a fairly consistent level of realized IC independent of universe size, ignoring any realistic limitations on manager information. Sample sizes were calibrated to attain the target IC levels for the largest sample size, and due to the positive skewness of the sample standard deviation in the denominator of the calculated IC, the smaller universe sizes tend to exhibit slightly smaller ICs. Realized ICs are displayed in Table 1. These statistics are the averages of the measured sample correlations between the estimates, in our case the sample means of simulated returns for 4, 13, and 30 periods, and their estimation targets, the referee mean vectors.

It should also be noted that unconstrained optimization will ideally benefit more than constrained optimization from greater information levels. In the 0.3 IC experiments, as the universe size increases, the unconstrained average maximum Sharpe ratio approaches the corresponding curve for the constrained cases. Although we do not present it here, there are paired values of IC and universe size where the unconstrained and constrained average maximum Sharpe ratio values will be equal, and for greater ICs or universe sizes, the unconstrained average maximum Sharpe ratio will exceed the corresponding value for the constrained cases. We know this because in the perfect information case, where the estimates equal the true parameters, the unconstrained maximum Sharpe ratio must exceed the constrained maximum for each optimization. Of course for the dataset presented here, the requirements for this crossover to occur are not realistically attainable for any real estimation procedure, but nevertheless it is an interesting study of the operating characteristics of constrained optimization versus unconstrained.

Larger universes, in our simulations, implicitly require larger levels of overall investment information from the manager, all other things the same. The slowly rising level of unconstrained average maximum Sharpe ratios as universe size increases is a necessary artifact of the simulation framework. In practice, adding assets is unlikely to add investment value beyond some optimal size universe consistent with the investor's level of information all other things the same. Indeed, beyond some optimal point, the unconstrained curve is likely to curve downward as the size of the optimization universe increases in applications. In practice, the accumulative costs associated with increasing the universe size will overcome the performance increases shown in our experiments, diminishing and eventually reversing performance gains. Increasing universe size with information of uncertain quality is never recommendable.

5.0 Summary and conclusions

Our discussions and simulation studies show that the popular four principles of optimized portfolio design based on applications of the Grinold formula – frequent trading, adding securities, adding forecast factors, and removing constraints – are not reliable and often self-defeating. The Grinold "law" does not help you invest better. Beyond some point the principles are perverse. The law does not tell you anything reliable you did not already know or should have known.

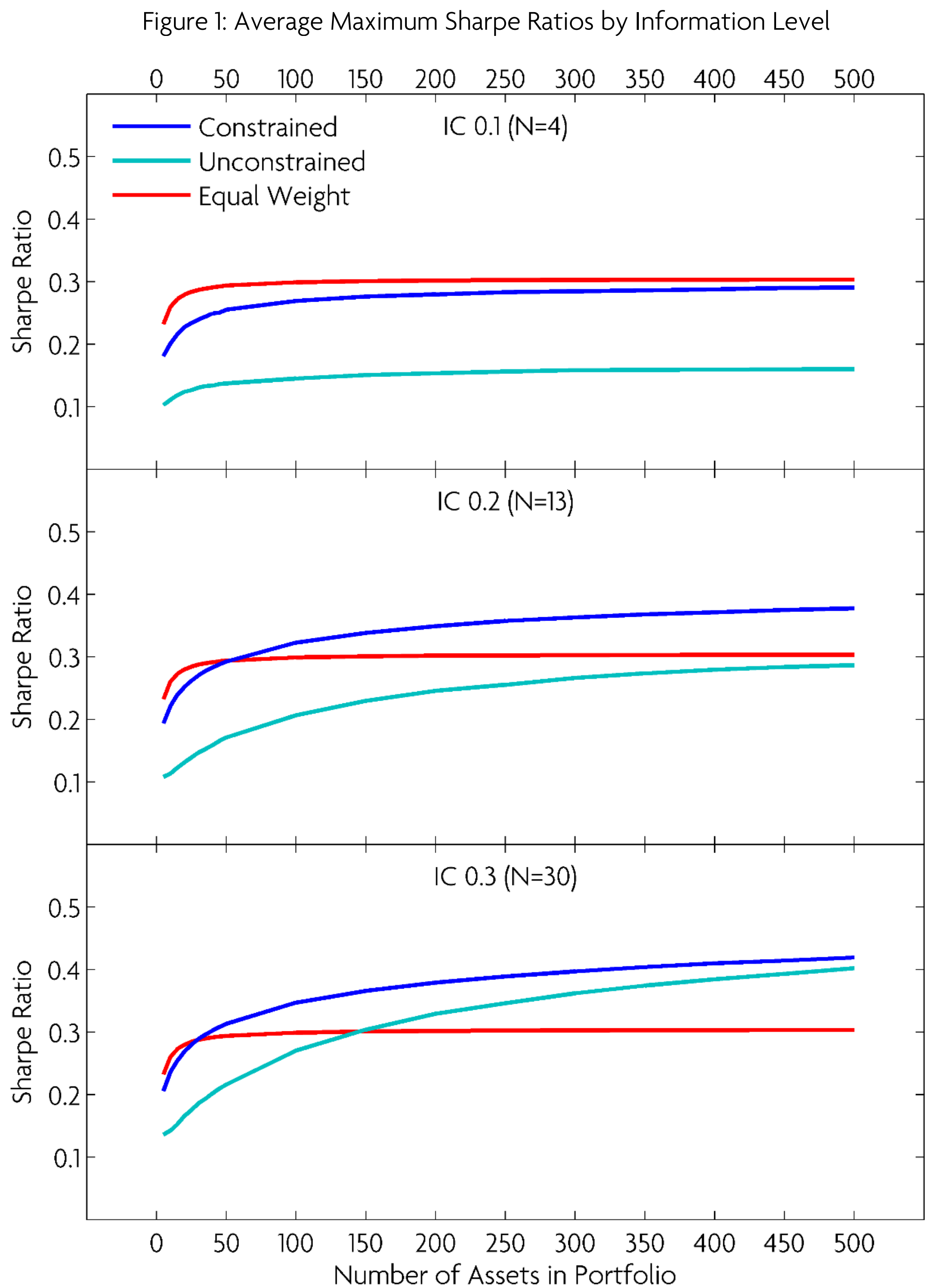

Figure 1: Average Max Sharpe Ratios for three different portfolio construction methods and three different information coefficients. Target information coefficients are not precisely attained by the simulations and realized ICs are shown below in Table 1. This experiment was run on many simulations of up to 500 U. S. stocks which had at least 20 years of contiguous monthly price data ending in December 2013.

Table 1: Realized IC for simulations from Figure 1 by universe size

IC	5	10	15	20	25	30	35	40	45	50
0.1 (N=4)	0.0982	0.1035	0.1075	0.0978	0.1038	0.1076	0.1129	0.1205	0.1119	0.1127
0.2 (N=13)	0.1810	0.1850	0.1882	0.1948	0.1964	0.1931	0.1980	0.1952	0.1978	0.2005
0.3 (N=30)	0.2597	0.2688	0.2797	0.2848	0.2883	0.2892	0.2876	0.2930	0.2912	0.2935
IC	100	150	200	250	300	350	400	450	500	
0.1 (N=4)	0.1138	0.1120	0.1117	0.1156	0.1140	0.1135	0.1134	0.1125	0.1146	
0.2 (N=13)	0.1996	0.2029	0.2029	0.2017	0.2039	0.2023	0.2010	0.2021	0.2045	
0.3 (N=30)	0.2984	0.3010	0.3020	0.3005	0.2998	0.2997	0.2984	0.3026	0.3019	

The Grinold square root law is a theoretical construct that failed to consider the impact of estimation error on MV portfolio optimization. None of our out-of-sample simulated average MSR curves reflect the availability of an out-of-sample square root improvement as a function of breadth even in our highly idealized and bend-over-backward framework tilted to advantage the Grinold application hypothesis. A nearly twenty-five year cohort of academic and practitioner research based on the formula and its extensions are not valid for practice. Rationales for investing in many hedge fund, long-short, and unconstrained strategies are likely invalid. The implications of applications of the Grinold law may adversely impact many hundreds of billions of dollars or more of assets under management in current practice.

The necessary conditions for reliably winning the investment game include: 1) investment significant information for assets in an optimization universe; 2) economically meaningful constraints; and 3) properly implemented estimation error sensitive portfolio optimization technology.

In the natural sciences, generally accepted laws are strongly supported by empirical evidence, and strictly specified so as to predict a measurable outcome. The fundamental law, as a statement of investment value, has little consensus as to the exact specifications of breadth or the measurability of IC, is not applicable to practice because it only concerns ex ante estimates, and has not been consistently supported by any empirical evidence, partly due to the difficulty of quantifying IR, IC, and BR. For these reasons it does not deserve the title of "Law" and should not be considered a foundational principle of any scientific approach to finance.

The roots of the failure of the Grinold formula applications go deep not only into twentieth century finance but more generally statistical science and beyond. It is the result of the use of analytical methods and the assumption of known probability distributions as in the certainty of a casino game. Grinold is simply one example in modern science of the fundamental and ubiquitous fallacy of regarding inference from in-sample statistics and fixed probability models as the full measure of uncertainty.[25]

A1: Appendix

Figure A1 shows the results from the simulation experiment using the referee's covariance as optimization inputs. We see that the impact of no estimation error in the covariance generally has a positive effect on the out-of-sample performance of the optimizations. For larger universe sizes,

[25] Weisberg op. cit., p. xiii.

the out-of-sample penalty in Sharpe ratio performance for covariance estimation under positivity constraints is greater than for the unconstrained case, for which it is negligible in this experiment. However, constrained optimization still dominates unconstrained out-of-sample in our experiment for universe sizes up to 500 assets for both covariance treatments. Using the referee covariance in this experiment forces attenuating performance gains to be attributed to the basic failure of the law, rather than anything to do with covariance estimation, and the inversion of the performance relationship claimed by CST is maintained. These results should not be surprising; nor do they represent any serious contradiction to our basic thesis that adding securities adds little if any investment value, all other things the same.

A2: Measuring Breadth

The ensemble of assets used for optimization in our experiment has non-zero correlations. A sample of size N from a correlated universe cannot attain the same precision for estimation as a similarly sized sample from a set of uncorrelated assets. Because of the way the estimates for mean and variance are constructed from asset-specific information, we are certain that each additional stock in the optimization is adding breadth, but exactly how much breadth is added is an important consideration for this research. In particular, we need to know whether breadth has a linear relation to the number of assets, so that the functional form of the performance can be compared against a square root law.

According to GK: "Breadth is defined as the number of independent forecasts of exceptional return we make per year." The question for our experiment is how many independent signals are in our forecast. Several approaches to answering this question are possible for different interpretations of the definition, from mathematically rigorous to practically important, i. e. significantly different from noise.

We first provide a mathematically rigorous answer to the question of breadth in our simulation experiment. Our forecast for the mean is the average of *N* observations from a random sample from a multivariate normal distribution with mean $\boldsymbol{\mu}$ and standard deviation $\boldsymbol{\Sigma}$. Thus the correlation of our forecasts is **R**, the same as the error distribution and the data. Buckle (2004) defines breadth mathematically as $\sum_{i,j} \pi_{ij} \rho^*_{ij}$, where π_{ij} is the correlation between the forecasts of assets *i* and *j*, and ρ^*_{ij} is the element in the i^{th} row and j^{th} column of the inverse of the forecast error correlation matrix. Since the forecast and forecast error correlation matrix are the same this sum is the same as the trace of $\mathbf{R}\,\mathbf{R}^{-1}$, identically equal to N, the number of assets. This characterization of the breadth makes intuitive sense since the covariance matrix of the data is full rank and the number of independent signals is *N*.

A principal components decomposition of our referee's covariance matrix confirms that none of the independent dimensions of the system vanish. All of the eigenvectors are needed to replicate our forecast to reasonable precision. If some of the eigenvalues were vanishingly small, the practical answer to the question of breadth would be quite different from the mathematically rigorous one. However, the full covariance matrix of 500 assets in our dataset has a smallest eigenvalue of over 10 basis points, which is likely significant for most definitions of statistical significance. This would correspond to an annualized standard deviation of approximately 11%, which is substantial by most measures. The submatrices of smaller portfolios tend to have even greater values for the smallest eigenvalue. This line of reasoning confirms that the effective breadth of a sample of size N from our universe is identically N in a practical sense as well as the theoretical one.

In our simulation experiment, breadth is clearly equal to the portfolio size. Each asset is adding a unit of stock-specific information to the system. Therefore the results of our simulation experiment hold as far as quantifying the functional form of the relationship between portfolios value as expressed by Sharpe ratio and breadth for our data.

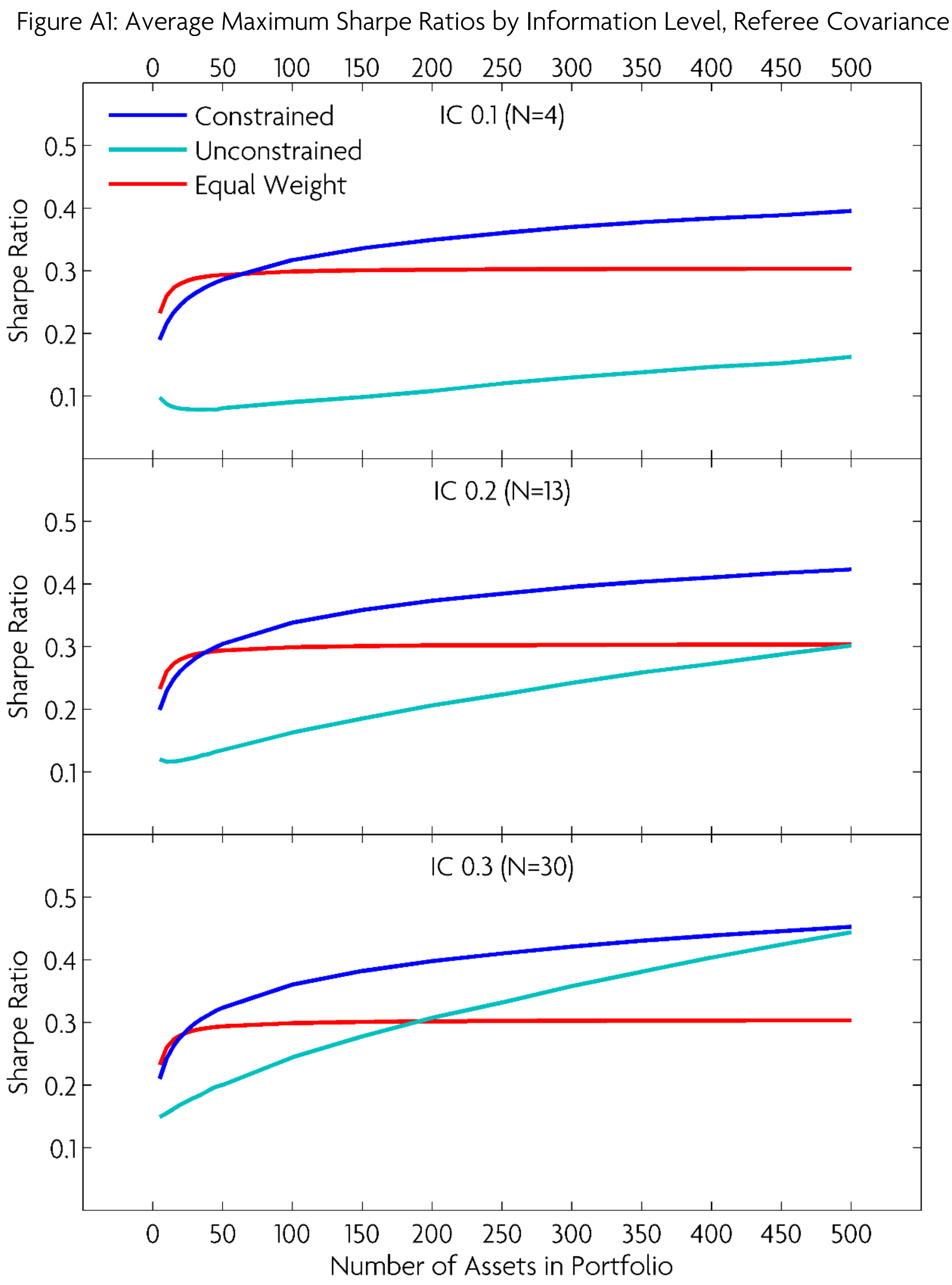

Figure A1: Average Sharpe Ratios for three different portfolio construction methods and three different information coefficients for the equity optimization case, using the referee's covariance matrix. Target information coefficients are not precisely attained by the simulations and realized ICs are the same as in Table 2, since the simulations were run with a random seed. This experiment was run on many simulations of up to 500 U. S. stocks which had at least 2 years of contiguous monthly price data ending in December 2013.

References

Anson, M., D. Chambers, K. Black, H. Kazemi. 2009. *An Introduction to Core Topics in Alternative Investments* CAIA Association, 2nd ed. 2012. Wiley.

Buckle, D. 2004. "How to Calculate Breadth: An Evolution of the Fundamental Law of Active Portfolio Management." *Journal of Asset Management* 4(6):393–405.

Clarke, R, H. deSilva, S. Thorley. 2002. "Portfolio Constraints and the Fundamental Law of Active Management." *Financial Analysts Journal* 58(5): 48-66.

Clarke, R., H. deSilva, S. Thorley. 2006. "The Fundamental Law of Active Portfolio Management." *Journal of Investment Management*, 4(3): 54-72.

Darnell, M. and K. Ferguson, 2014. "Thoughts On Grinold & Kahn's Fundamental Law Of Active Management" First Quadrant Perspective..

Diderich, C. 2009. *Positive Alpha Generation: Designing Sound Investment Processes*, Wiley.

Ding, Z. 2010. "The Fundamental Law of Active Management: Time Series Dynamics and Cross-Sectional Properties." Russell and Thaler Asset Management working paper.

Farrell, J. 1983. *Guide to Portfolio Management*. Wiley, NY.

Focardi, S. and F. Fabozzi, 2004. The Mathematics for Financial Modeling and Asset Management, Wiley.

Frost, P. and J. Savarino. 1988. "For Better Performance: Constrain Portfolio Weights." *Journal of Portfolio Management* 15(1):29-34.

Gorman, L., S. Sapra,, R. Weigand. 2010. "The role of cross-sectional dispersion in active portfolio management." *Investment Management and Financial Innovations*, 7(3):54-64.

Grinold, R. 1989. "The Fundamental Law of Active Management." *Journal of Portfolio Management* 15(3): 30-37.

Grinold, R. and R. Kahn. 1995. *Active Portfolio Management: Quantitative Theory and Applications*. Chicago: Irwin.

Grinold, R. and R. Kahn, 1999. *Active Portfolio Management: A Quantitative Approach for Providing Superior Returns and Controlling Risk*. 2nd ed. New York: McGraw-Hill.

Harvey, C., R. Liechty, R. Liechty 2008. "Bayes vs. Resampled efficiency: A Rematch." *Journal of Investment Management* 6(1):29-45.

Harvey, C., R. Liechty, R. Liechty 2008b. "A Response to Richard Michaud and Robert Michaud: Letter to the Editor." *Journal Of Investment Management*. 6(3): 114.

Huij, J. and J. Derwall. 2011. "Global Equity Fund Performance, Portfolio Concentration, and the Fundamental Law of Active Management." *Journal of Banking & Finance*, 35(1):155-165.

Jacobs, B. and K. Levy, 2008. *Investment Management: An Architecture for the Equity Market*. Wiley

Jobson, D. and B. Korkie, 1981. "Putting Markowitz Theory to Work." *Journal of Portfolio Management* 7(4): 70-74.

Kahn, R. 1997. "The Fundamental Law of Active Management: Seven Quantitative Insights into Active Management—Part 3," BARRA Newsletter Winter.

Kroll, B. D. Trichilo, and J. Braun, 2005. "Extending the Fundamental Law of Investment Management (or How I Learned to Stop Worrying and Love Shorting Stocks)." JP Morgan Asset Management white paper.

Ledoit, O., and Wolf, M. 2004a. "A Well-Conditioned Estimator for Large-Dimensional Covariance Matrices". *Journal of Multivariate Analysis*, 88(2): 365-411.

Markowitz, H. 1952. "Portfolio Selection." *Journal of Finance* 7(1): 77-91.

Markowitz, H., 1959. *Portfolio Selection: Efficient Diversification of Investments*. New York: John Wiley and Sons. 1991. 2nd ed. Cambridge, MA: Basil Blackwell.

Markowitz, H. 2005. "Market Efficiency: A Theoretical Distinction and So What?" *Financial Analysts Journal* 61(5): 17-30.

Markowitz, H. and N. Usmen, 2003. "Diffuse Priors vs. Resampled Frontiers: An Experiment." *Journal Of Investment Management* 1(4): 9-25.

Menchero, J. 2017. "Rethinking the Fundamental Law of Active Management." *Journal Of Investment Management* 15(2):92-107.

Merton, R. 1987. "Presidential Address: A Simple Model of Capital Market Equilibrium with Incomplete Information." *Journal of Finance* 42(3): 483-510.

Michaud, R. 1989. "The Markowitz Optimization Enigma: Is Optimized Optimal?" *Financial Analyst Journal* 45(1):31-42.

Michaud, R. 1990. "Demystifying Multiple Valuation Models." *Financial Analysts Journal* 46(1):6-8.

Michaud, R. 1993. "Are Long-Short Equity Strategies Superior?" *Financial Analyst Journal* 49(6):44-49.

Michaud, R., 1998. *Efficient Asset Management*. New York: Oxford University Press, 2001. First published by Harvard Business School Press.

Michaud, R. 1999. *Investment Styles, Market Anomalies, and Global Stock Selection*. Research Foundation of the Chartered Financial Institute, Charlottesville.

Michaud, R. and R. Michaud. 2008a. *Efficient Asset Management*. 2nd ed. Oxford University Press: New York.

Michaud, R. and R. Michaud 2008b. "Estimation Error and Portfolio Optimization: A Resampling Solution." *Journal Of Investment Management* 6(1):8-28.

Michaud, R. and R. Michaud 2008c. "Discussion on article by Campbell R. Harvey, John C. Liechty and Merill W. Liechty, Bayes vs. Resampling: A Rematch (2008, Volume 6, Number 1), *Journal Of Investment Management* 6(3):1-2.

Michaud, R., D. Esch, and R. Michaud 2012. "Portfolio Monitoring in Theory and Practice," *Journal Of Investment Management* 10(4):5-18.

Qian, E and R. Hua., 2004. "Active Risk and Information Ratio." *Journal of Investment Management*, 2(3):1-15.

Roll, R. 1992. "A Mean/Variance Analysis of Tracking Error." *The Journal of Portfolio Management*, 18(4):13-22.

Schulmerich, M., Y. Leporcher, and C. Eu. 2015. *Applied Asset and Risk Management,* Springer: Heidelberg

Utermann, A. 2013. "The Changing Nature of Equity Markets and the Need for More Active Management" Allianz GI Global Capital Markets & Thematic Research.

Weisberg, H. 2014. *Willful Ignorance: The Mismeasure of Uncertainty*. Wiley: New York.

Zhou, G. 2008. "On the Fundamental Law of Active Portfolio Management: What Happens If Our Estimates Are Wrong?" *Journal of Portfolio Management,* 34(4):26-3