BEST ROTATED APPROXIMATION

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1. INTRODUCTION

The central theme of this paper is the minimax approximation of a function f rotated by an angle α about the origin and the study of the angle of rotation, α^*, of least minimax error. However, many of our results, particularly existence theorems, will be valid for any choice of norm. In the following formulation of our problem, we will want to include, with obvious modifications, the cases when f is either single-valued or multi-valued and defined on a finite subset of [0,1].

Let f(x) be an arbitrary given continuous real-valued function on [0,1]. Let $G = \{g(A,x)\}$ be a given set of admissible approximations and let $||\cdot||$ be an appropriate norm on [0,1]. Then a best approximation (B.A.) to f is a $g^* = g(A^*,x)$ belonging to G such that

$$||f(x) - g(A^*,x)|| \leq ||f(x) - g(A,x)||$$

for all $g \in G$. Here A stands for the real parameters $a_1, a_2, \ldots a_n$ of the approximating function g. If $||\cdot||$ is the uniform or L_∞ norm on [0,1] then g^* is said to be a minimax approximation to f.

The class of approximations which we shall consider is the unisolvent class of some fixed degree n.

<u>Definition 1.1</u>: The class G_n of n-parameter continuous real-valued functions is said to be unisolvent of degree n on [a,b], if for any

distinct points $x_1,\ldots,x_n$ of $[a,b]$ and any real numbers $y_1,\ldots,y_n$ there is a unique $g(A,x) \in G_n$ such that the equations

$$g(A,x_i) = y_i, \qquad i = 1,\ldots,n$$

are satisfied.

We will wish to consider two special cases of unisolvent classes of functions of degree n: The class L_n of linear combinations of functions satisfying the Chebychev set property or generalized polynomial approximations [10,p.55] of the form: $\sum_{i=1}^{n} a_i\phi_i(x)$; and P_{n-1}, the polynomial class of degree n-1 or less. The class P_{n-1} has the additional property of being unisolvent over any closed bounded interval.

Let $f(t)$ be a continuous real-valued function on $[0,1]$. We will use $t \in [0,1]$ to parametrize this function after rotation. Before rotation the function is defined by $x=t$, $y=f(t)$. After rotation by an angle α (which we will measure positive clockwise), the curve is given by the parametric equations

$$\begin{aligned} x &= x_\alpha(t) = t\cos\alpha + f(t)\sin\alpha \\ y &= y_\alpha(t) = -t\sin\alpha + f(t)\cos\alpha \end{aligned} \tag{1.1}$$

where t ranges from 0 to 1. We will refer to the curve (1.1) as the result of an "α-rotation" of f. If $x_\alpha(t)$ is monotone, the y given by (1.1) is a function of x; otherwise it is not;

i.e., the curve has become multi-valued.

The range of abscissa values of the α-rotated f is the range of the continuous functions $x_\alpha(t)$, which we will denote as $[a,b]$. Thus if we wish to exploit unisolvence, the class of approximations must be unisolvent over the interval $[a,b]$.

Now consider an approximation $g(A,x) \varepsilon G_n$ to the α-rotated f. Parametrized by t, the error

$$e_\alpha(t) = y_\alpha(t) - g(A,x_\alpha(t)), \; t \varepsilon [0,1] \tag{1.2}$$

is a continuous function in t and periodic with period 2π in α. If $G_n = L_n$ then (1.2) becomes

$$e_\alpha(t) = y_\alpha(t) - \sum_{i=1}^{n} a_i \phi_i(x_\alpha(t)), \; t \varepsilon [0,1] \tag{1.3}$$

and if $G_n = P_{n-1}$ then (1.2) is of the form

$$e_\alpha(t) = y_\alpha(t) - \sum_{i=0}^{n-1} a_1 (x_\alpha(t))^i, \; t \varepsilon [0,1]. \tag{1.4}$$

We make the following definition for the B.A. error for an α-rotation of f.

<u>Definition 1.2</u>: Given a unisolvent class of functions G_n, a continuous function f on $[0,1]$, and a given rotation α, then

$e^*(\alpha)$ is the error of the α-rotation best approximation; i.e.,

$$e^*(\alpha) = ||y_\alpha(t) - g(A^*, x_\alpha(t))|| \ .$$

When $||\cdot||$ is the uniform norm, then $e^*(\alpha)$ is the minimax error for an α-rotation of f.

The α-rotation minimax approximation problem is a generalization of [4] and has been solved in [8]. In order to state the results we will need the following definitions:

<u>Definition 1.3</u>: An extremum of $e_\alpha(t)$ is a point $t_o \in [0,1]$ such that $e_\alpha(t_o) = y_\alpha(t_o) - g(A, x_\alpha(t_o)) = \pm ||e_\alpha(t)||$.

<u>Definition 1.4</u>: x_o is a straddle point of $e_\alpha(t)$ if there exists t_1 and $t_2 \in [0,1]$, $t_1 \neq t_2$, $x_o = x_\alpha(t_1) = x_\alpha(t_2)$, and $e_\alpha(t_1) = -e_\alpha(t_2) = \pm ||e_\alpha(t)||$.

<u>Definition 1.5</u>: $e_\alpha(t)$ is said to have n+1 equioscillating extrema or critical points if there exists n+1 extrema $t_1, \ldots, t_{n+1}$ of $e_\alpha(t)$ such that for some reordering: t_{i_j}

$$x_\alpha(t_{i_1}) < x_\alpha(t_{i_2}) < \ldots < x_\alpha(t_{i_{n+1}}) \text{ and } e_\alpha(t_{i_j}) = (-1)^j(\pm||e_\alpha(t)||.$$

We now state existence, characterization, and uniqueness results for α-rotation minimax approximation [8].

<u>Theorem 1.6</u> (Existence): The minimax approximation $g^* \varepsilon G_n$, of the α-rotation of a continuous function, exists.

<u>Theorem 1.7</u> (Characterization): $g^* = g(A^*, x_\alpha(t)) \varepsilon G_n$ is a minimax approximation to an α-rotation of f if and only if

a) $e_\alpha^*(t) = y_\alpha(t) - g(A^*, x_\alpha(t))$ has a straddle point or

b) $e_\alpha^*(t)$ has n+1 equioscillating extrema.

<u>Theorem 1.8</u> (Uniqueness): If $e_\alpha^*(t)$ has n+1 equioscillating extrema then g^* is unique.

An obvious simplification of Theorems 1.7 and 1.8 occurs when the α-rotation of f is a function. Accordingly we make the following

<u>Definition 1.9</u>: $R = \{ \alpha | \text{the } \alpha\text{-rotation of f is a function} \}$.

<u>Theorem 1.10</u>: If $\alpha \varepsilon R$, then α-rotation minimax approximation exists, is unique, and is characterized by the condition that the error function have n+1 equioscillating extrema.

We note that when $\alpha \varepsilon R$, $x_\alpha(t)$ is a strict monotone function of t and hence if $t_1 < \ldots < t_{n+1}$, the reordering of Definition 1.5 is unnecessary.

Our primary concern will be with optimizing α; i.e., with finding α^* for which $e^*(\alpha)$ is least.

<u>Definition 1.11:</u> A best rotated best approximation, or, more simply, a best rotated approximation (B.R.A.) is a $g^*_{\alpha^*} \varepsilon G_n$ such that $g^*_{\alpha^*}$ is a best approximation for the α^*-rotation of f and

$$e^*(\alpha^*) \leq e^*(\alpha) \quad \text{for } \alpha \varepsilon [0,2\pi] .$$

When $||\cdot||$ is the uniform norm, then we will call $g^*_{\alpha^*}$ a best rotated minimax approximation (B.R.M.A.).

It is easy to see that $e^*(\alpha)$ is periodic with period π for polynomial approximation.

We illustrate the concepts introduced with the following simple example. The function $\sqrt{x}$ on [0,1] is well known to be a rather difficult function to best approximate. If the function is rotated clockwise, by 90°, about the origin, second degree polynomial best approximation is exact.

A third degree polynomial approximation has the same number of free parameters as a second degree polynomial best rotated approximation. However, our example demonstrates that a second degree polynomial B.R.A. may have smaller B.A. error than third (or higher) degree polynomial B.A. Of course, the opposite case is possible as can be seen with the function $x^3, x \varepsilon [0,1]$.

There are numerous applications where one wishes to find best approximations to functions which are to be considered simply as curves in the plane and where a geometric distance is more appropriate than function distance as a measure of the error. Such

approximations are more aptly termed "curve fits" than approximations. Evidently, a B.R.A. is a curve fit to f.

Since B.A. error for polynomial approximation is invariant under translation of the function, a best rotated polynomial approximation is a polynomial best approximation with least B.A. error over all rigid transformations of f.

The curve fit obtained by a draftsman using a french curve to fair data is usually made up of a sequence of curves, drawn from the french curve, which have been pieced together in a smooth or spline-like fashion. Each section of the resulting smooth curve is an optimal curve fit under all transformations of the french curve or, equivalently, under all rigid transformations of the data, subject to continuity constraints at the end points. If, however, we consider only one curve, drawn from the french curve, as our curve fit, and thus dispense with continuity constraints at the end points, the resulting approximation is similar to a polynomial B.R.A. in that both are optimal approximations under all rigid transformations of the data. As a result, a polynomial B.R.A. can be considered as a partial answer to the polynomial "Automatic French Curve" problem. (See [3] for further discussion of the "Automatic French Curve").

There are two useful interpretations of the best rotated approximation problem. By rotating f and searching for α^*, we are attempting to increase the approximability of f, to orient f in an optimal fashion for best approximation. Our solution, in this case, would then be to rotate f by means of (1.1) and use the B.R.A. as our B.A. to the α^*-rotation of f.

Alternatively, we can view the introduction of the rotation parameter as enhancing the curve fitting power of the given class of approximations. Hence, our solution would then be to rotate the B.R.A. by $-\alpha^*$ using (1.1) and the $-\alpha^*$-rotation of the B.R.A. is our curve fit to (unrotated) f. It is of interest to note that in many cases our B.R.M.A. curve fit to (unrotated) f is a better approximation, considering function distance as the error of approximation, than the minimax approximation to f (see [8, Section 5.4]).

An example where a B.R.A. may be appropriate is the following: If we wished to fair points $f(x_i)$, $i = 1,\ldots,N$, given as design specifications for a cross section of some physical surface, it is often the case that there is no a priori reason to prefer one orientation over another. If accuracy of approximation is the only factor to be considered, then we can dispense with questions of optimal orientations of the data by allowing sufficiently high degree polynomial approximations. But for many applications, particularly in computer-aided design, a curve fairing process requires the approximation to be relatively free of extraneous changes in direction or "wiggles". In this regard, a polynomial B.R.A. of degree n would, in general, have one less "wiggle" than a B.A. of degree n+1. The need for "wiggle-free" approximations has been, in part, responsible for proscribing the application of minimax theory to automated curve fitting.

It is possible that for a given function and class of approximations, there does not exist $\alpha^* \varepsilon R$. For most purposes, an approximation where $\alpha^* \notin R$ would not be useful. However, we may consider this result to be due to the fact that our class of approximations was not of sufficiently high degree and did not sufficiently resemble the given function. The following result for B.R.M.A. guarantees that all $\alpha^* \varepsilon R$ when the class of approximations is of sufficiently high degree.

We will require the concept of a fundamental Chebychev set.

<u>Definition 1.12</u>: The set $\{\phi_k\}$ of the Chebychev approximating class L_n is said to form a fundamental Chebychev set if each element of $D[0,1]$ can be arbitrarily well approximated by linear combinations of elements of the set $\{\phi_k\}$ [1,p.87].

<u>Theorem 1.13</u>: For B.R.M.A., if the set of functions $\{\phi_k\}$ of the Chebychev approximating class L_n forms a fundamental Chebychev set, then all $\alpha^* \varepsilon R$, for n sufficiently large.

<u>Proof</u>: Suppose $\exists$ an $\alpha_o^* \notin R$ $\forall$ $n > N$. Since $\exists$ $x_o \varepsilon [a,b]$ for which the α_o^*-rotation of f is, at least, bi-valued, the best any α_o^*-rotation minimax approximation can do, for any degree, is halve the error at the discontinuity. But by the fundamental Chebychev set property, $\exists$ $\alpha' \varepsilon R$ and N' such that $\forall$ $n > N'$, the α'-rotation minimax error is less than at α_o^*.

Finally, we note that B.R.A. is rotation-independent in the sense that neither the approximation $g^*_{\alpha^*}$ nor the value of the error $e^*(\alpha^*)$ depend on the given orientation of the function.

The problem of characterizing α^* has been extensively studied in the case of straight line approximation for the discrete ℓ_2 norm. The best approximating line for α^* is called the orthogonal regression line [7]. Roos [11] has given a solution for a best approximating straight line in the ℓ_2 norm which is invariant to rotation, translation, and linear stretching. Michaud [9] has given a solution for α^* for straight line approximation in the L_2 norm.

In Chapter 2 we show that every continuous function on [0,1] or any subset thereof, possesses a B.R.A. for the Chebychev class of approximations. The proof is a consequence of our result that $e^*(\alpha)$ is a continuous function of α. We apply some results due to Curtis and Powell [2] to obtain necessity conditions for the best rotated minimax approximation. We then prove that α^*, for B.R.M.A., is not in general unique.

In Chapter 3 we present an algorithm for computing α^* and the polynomial B.R.M.A. We then give some numerical results obtained using the algorithm and a parameter search technique. For most of the cases examined, the error function of the B.R.M.A. had one more equioscillating extrema than necessary, coinciding with those instances for which the effect of the rotation parameter on the minimax error was most significant. However, we give examples which demonstrate that the error function of a B.R.M.A. cannot, in general, be characterized by n+2 equioscillating extrema.

2. BEST ROTATED APPROXIMATION: THEORETICAL RESULTS

2.1 EXISTENCE

Unless otherwise stated, the approximating class in this chapter will be assumed to be L_n. We will make use of the following

Lemma 2.1: If $f(x) \neq a_1 x + a_o$, then the length of the interval $[a_\alpha, b_\alpha]$, the range of $x_\alpha(t)$ of the α-rotation of f, is never zero, for all $\alpha \in [0,\pi]$.

Proof: Let $M(\alpha) = b_\alpha - a_\alpha$, be the length of the interval of the range of $x_\alpha(t)$. Let $u = \inf_\alpha M(\alpha)$. $M(\alpha)$ is continuous and bounded on $[0,\pi]$ and hence achieves its infimum. If $u = 0$, then $\exists\ \alpha'$ such that

$$x_{\alpha'}(t) = t\cos\alpha' + f(t)\sin\alpha' = c \quad .$$

Now $\sin\alpha = 0$ when $\alpha = 0$ or π and hence when $M(\alpha)$ is not zero. Thus at α',

$$f(t) = -c/\sin\alpha' + t\cot\alpha' \quad .$$

But this case is ruled out by hypothesis. Therefore $u \neq 0$.

By [6, Ch. 7, Thm. 1] we have the result that α-rotated B.A. exists and hence $e^*(\alpha)$ is defined for all $\alpha \in [0,2\pi]$, for the

class of Chebychev approximations, or, more generally, for linear combinations of n linearly independent functions. We will now prove that $e^*(\alpha)$ is continuous.

<u>Theorem 2.2</u>: If $f(x) \neq a_1 x + a_o$ then $e^*(\alpha)$ is a continuous real-valued function of α.

<u>Proof</u>: The proof is indirect. Suppose $e^*(\alpha)$ is not continuous; then $\exists\ \alpha_o \in [0,2\pi]$ such that $e^*(\alpha)$ is not continuous at α_o. Then $\exists\ \varepsilon > 0$ such that in every δ-neighborhood of $\alpha_o\ \exists\ \alpha$ such that $|e^*(\alpha) - e^*(\alpha_o)| \geq \varepsilon$.

We discuss separately the Cases A and B.

<u>Case A</u>: $|e^*(\alpha) - e^*(\alpha_o)| = e^*(\alpha) - e^*(\alpha_o) =$

$$||y_\alpha(t) - \Sigma\, a_k^\alpha\, \phi_k(x_\alpha(t))|| - ||y_{\alpha_o}(t) - \Sigma\, a_k^{\alpha_o}\, \phi_k(x_{\alpha_o}(t))||$$

where a_k^α denotes the kth coefficient of the Chebychev best approximation of degree n for the α-rotation of f.

Since $||y_\alpha(t) - \Sigma\, a_k^\alpha\, \phi_k(x_\alpha(t))|| \leq ||y_\alpha(t) - \Sigma\, a_k^{\alpha_o}\, \phi_k(x_\alpha(t))||$ then $e^*(\alpha) - e^*(\alpha_o)$

$$\leq ||y_\alpha(t) - \Sigma\, a_k^{\alpha_o}\, \phi_k(x_\alpha(t))|| - ||y_{\alpha_o}(t) - \Sigma\, a_k^{\alpha_o}\, \phi_k(x_{\alpha_o}(t))||$$

$$\leq ||y_\alpha(t) - y_{\alpha_o}(t)|| + ||\Sigma\, a_k^{\alpha_o}\, (\phi_k(x_\alpha(t)) - \phi_k(x_{\alpha_o}(t)))||$$

$$\leq ||y_\alpha(t) - y_{\alpha_o}(t)|| + \Sigma |a_k^{\alpha_o}|\, ||\phi_k(x_\alpha(t)) - \phi_k(x_{\alpha_o}(t))||. \qquad (2.1)$$

Since

$$||\Sigma\, a_k^{\alpha_o}\, \phi_k(x_{\alpha_o}(t))|| \leq 2||y_{\alpha_o}(t)||$$

the coefficients of $\Sigma\, a_k^{\alpha_o}\, \phi_k(x_\alpha(t))$ are bounded [10,pp.24-25]. For sufficiently small $\delta > 0$, since $y_\alpha(t)$ and $x_\alpha(t)$ are continuous functions of α, and the ϕ_k's are continuous functions, (2.1) can be made less than ε. This is a contradiction.

Case B: From Case A we can immediately conclude a similar result to (2.1); i.e.,

$$e^*(\alpha_o) - e^*(\alpha) \leq ||y_{\alpha_o}(t) - y_\alpha(t)|| + \Sigma|a_k^\alpha|\,||\phi_k(x_\alpha(t)) - \phi_k(x_{\alpha_o}(t))||. \quad (2.2)$$

Since for all α, $||y_\alpha(t)||$ is bounded, then $\exists$ a constant N such that

$$||\Sigma\, a_k^\alpha\, \phi_k(x_\alpha(t))|| \leq 2||y_\alpha(t)|| \leq N, \text{ for all } \alpha \in [0,2\pi]\,. \quad (2.3)$$

Let

$$\Phi(t,a_k) = \Sigma\, a_k^\alpha\, \phi_k(x_\alpha(t))$$

be a best approximation of $y_\alpha(t)$. Hence the $\{a_k^\alpha\}$ satisfy (2.3). If the $\{a_k^\alpha\}$ are unbounded, then $\exists$ a K and a sequence $\alpha_1,\ldots,\alpha_j,\ldots$

such that $a_K^{\alpha_j}$ is unbounded; i.e., $|a_K^{\alpha_j}| \to \infty$, $j \to \infty$.

We can then choose a subsequence of α_j such that

$$\max_k |a_k^{\alpha_j}| = |a_K^{\alpha_j}| \tag{2.4}$$

Finally, since $\alpha \in [0,2\pi]$, we choose a subsequence of α_j satisfying (2.4) such that $\alpha_j \to \alpha'$.

Now consider the sequence

$$\psi_j(t,b_k) = \Sigma(a_k^{\alpha_j}/a_K^{\alpha_j})\phi_k(x_\alpha(t)) = \Phi(t,a_k^{\alpha_j})/a_K^{\alpha_j} \to 0, \ j \to \infty . \tag{2.5}$$

Since the magnitudes of $|a_k^{\alpha_j}/a_K^{\alpha_j}|$ are all bounded by unity, we can extract a subsequence converging to b_k', and since (2.5) is a continuous function defined on a compact set then

$$\phi_K(x_{\alpha'}(t) + \sum_{\substack{k=0 \\ k\neq K}}^{n} b_k' \phi_k(x_{\alpha'}(t)) = 0 \quad . \tag{2.6}$$

Equation (2.6) is defined on some non-zero interval by Lemma 2.1. But this contradicts the unisolvence of (2.6). Hence a_k^α must be bounded for all α.

Therefore, for sufficiently small $\delta > 0$, (2.2) can be made less than ε, which is a contradiction.

<u>Proposition 2.3</u>: $e^*(\alpha)$ is a bounded continuous real-valued function periodic with period 2π, provided $f(x) \neq ax + b$, f continuous on $[0,1]$.

<u>Corollary 2.4</u> (Existence): The optimal orientation α^* and the best rotated approximation $g^*_{\alpha^*} \varepsilon L_n$ exists for continuous f defined on [0,1], provided $f(x) \neq ax + b$.

<u>Corollary 2.5</u> (Existence): If one of the ϕ_k's of the L_n approximating class is a non-zero constant, then α^* and $g^*_{\alpha^*}$ exists for all continuous f(x) defined on [0,1].

<u>Proof</u>: If $f(x) = ax + b$, there exists α^* such that $e^*(\alpha^*) = 0$ since $\exists$ a rotation for which a constant approximates f(x) exactly.

From Corollary 2.5 we can conclude that polynomial B.R.A. exists for all $f \varepsilon C[0,1]$.

We note that if the functions ϕ_k, $k = 1,\ldots,n$, are linearly independent, the proof of Theorem 2.2 is unchanged and hence all the preceding results of this section are valid.

<u>Corollary 2.6</u>: If f is defined on a finite point set or any subset $\{t_i\}$ of [0,1] such that the cardinality of the set $\{x_\alpha(t_i)\}$ is greater than or equal to n for all $\alpha \varepsilon [0, \pi]$, then $e^*(\alpha)$ is continuous on $[0,2\pi]$.

<u>Proof</u>: The condition on the cardinality of the set $\{x_\alpha(t_i)\}$ guarantees that there will be enough points at any α, and hence at α' of the proof of Theorem 2.2 so that the contradiction following from equation (2.6) holds.

<u>Corollary 2.7</u> (Existence, discrete case): If the cardinality of the set $\{x_\alpha(t_i)\}$ is greater than or equal to n for all $\alpha \in [0,\pi]$, then α^* and $g^*_{\alpha^*} \in L_n$ exists for f defined on any subset $\{t_i\}$ of $[0,1]$.

In the computation of α^* and the B.R.M.A. $g^*_{\alpha^*}$, we will often find it desirable to replace the interval $[0,1]$ by a finite point set and seek an approximation which is optimum on that set. The following result due to Cheney [1, p. 86] relates the continuous and discrete α-rotation minimax error. We will need to establish some notation. Let X_α be the range of $x_\alpha(t)$ and Y_α be the subset of points $x_\alpha(t_i) \in X_\alpha$ for $\alpha \in [0,\pi]$. We define

$$|Y_\alpha| = \max_{x \in X_\alpha} \inf_{y \in Y_\alpha} |x - y| \quad .$$

<u>Theorem 2.8</u>: $e^*(\alpha)|_{Y_\alpha} \longrightarrow e^*(\alpha)|_{X_\alpha}$ as $|Y_\alpha| \longrightarrow 0$. Evidently $e^*(\alpha)|_{Y_\alpha} \leq e^*(\alpha)|_{X_\alpha}$. Let α^*_d and α^*_c denote respectively the optimal rotation for f defined over the point set $\{t_i\}$ and $[0,1]$. Then $e^*(\alpha^*_d) \leq e^*(\alpha^*_c)$ and $e^*(\alpha^*_d) \longrightarrow e^*(\alpha^*_c)$ as $\sup_\alpha |Y_\alpha| \longrightarrow 0$. If α^*_c is unique, then $\alpha^*_d \longrightarrow \alpha^*_c$ as $\sup_\alpha |Y_\alpha| \longrightarrow 0$.

2.2 CHARACTERIZATION OF BEST ROTATED MINIMAX APPROXIMATION

2.2.1. THE CURTIS - POWELL THEOREMS

The goal of this section is to apply some results in the theory of non-linear minimax approximation due to Curtis and Powell [2] to characterizing Chebychev and polynomial best rotated minimax approximations where we assume $f \in C^1[0,1]$. For the L_n class of approximations we will need to assume that the ϕ_k's have continuous first derivatives on $[a,b]$. $\Phi(t,\lambda^*) = \Phi(t,\lambda_1^*,\ldots,\lambda_{n+1}^*)$ is a minimax approximation to $f(t)$ if the parameters λ_i^* are such that

$$\max_{t\in[0,1]} |f(t) - \Phi(t, \lambda_1^*,\ldots,\lambda_{n+1}^*)| \tag{2.7}$$

is minimized.

The minimax approximation of $e_\alpha(t)$ is not in the form (2.7) of the Curtis and Powell (C-P) problem. However, $e_\alpha(t)$ can easily be rewritten to fit the C-P problem.

$$e_\alpha(t) = f(t) - \Phi(t,\lambda) = f(t) - (f(t) - y_\alpha(t) + \Sigma a_k \phi_k(x_\alpha(t)) \quad t \in [0,1] \tag{2.8}$$

where the C-P approximating function is

$$\Phi(t,\lambda) = f(t) - y_\alpha(t) + \Sigma a_k \phi_k(x_\alpha(t)), \quad t \in [0,1] \tag{2.9}$$

and the C-P parameters are

$$\lambda_1 = a_1,\ \lambda_2 = a_2, \ldots, \lambda_n = a_n, \lambda_{n+1} = \alpha.$$

<u>Proposition 2.9</u>: Let $\Phi(t,\lambda)$ be the approximating function (2.9). Let $\Phi(t,\lambda^*)$ be the minimax approximation defined by (2.7) Then a C-P minimax approximation is a best rotated minimax approximation where $\lambda_i^* = a_i^*$ and $\lambda_{n+1}^* = \alpha^*$.

<u>Proof</u>: We denote the minimax error of (2.7) as $E(\lambda^*)$.
Evidently $E(\lambda^*) \leq e^*(\alpha^*)$.
Given $\lambda_{n+1}^* = \alpha$, if $\lambda_i^* \neq a_i^*$ then $E(\lambda^*)$ is not minimax. If $\lambda_{n+1}^* \neq \alpha^*$ then by Definition 1.11 $E(\lambda^*)$ is not minimax.

<u>Definition 2.10</u>: Let r equal the number of extrema of $e^*_{\alpha^*}(t)$ where $\Phi(t,\lambda^*) = \Phi(t,a_1^*,a_2^*,\ldots,a_n^*,\alpha^*)$. We define an r x n+1 matrix whose elements are given by

$$D_{ij} = \left.\frac{\partial \Phi(t_i,\lambda)}{\partial \lambda_j}\right|_{\lambda=\lambda^*} \tag{2.10}$$

If h^* is the B.R.M.A. error then s_i is defined as the sign of the error $e^*_{\alpha^*}(t)$ at the extremum t_i

$$e^*_{\alpha^*}(t_i) = s_i h^*, \quad i = 1,\ldots,r.$$

In the event that $r = n + 2$, there are $n + 2$ square matrices of order $n + 1$, denoted by $\Delta_1, \ldots, \Delta_r$, where Δ_k is the matrix obtained by deleting the kth row of the matrix D. p_k is reserved for the determinant of Δ_k multiplied by $(-1)^k$.

<u>Theorem 2.11</u> (Curtis and Powell): At λ^* the rank of D is less than r.

<u>Theorem 2.12</u> (Curtis and Powell): At λ^*, if $r = n + 2$, the signs of $s_1, \ldots, s_r$ are all the same as or all opposite to the signs of $p_1, \ldots, p_r$.

An examination of the proof of the Curtis and Powell theorems reveals that the C-P conditions are necessary at a relative minimum and hence must be satisfied at every relative minimum. For our purposes, the C-P theorems are tests, on the basis of which consideration can be narrowed to those approximations which satisfy the conditions.

By (2.10) the C-P matrix is:

$$D = \begin{pmatrix} \phi_1(x_\alpha(t_1))\phi_2(x_\alpha(t_1))\ldots & \phi_n(x_\alpha(t)) & y_\alpha(t_1)(\Sigma a_k \phi_k'(x_\alpha(t_1)))+x_\alpha(t_1) \\ \phi_1(x_\alpha(t_2))\phi_2(x_\alpha(t_2))\ldots & \phi_n(x_\alpha(t_2)) & y_\alpha(t_2)(\Sigma a_k \phi_k'(x_\alpha(t_2)))+x_\alpha(t_2) \\ \vdots & & \\ \phi_1(x_\alpha(t_r))\phi_2(x_\alpha(t_r))\ldots & \phi_n(x_\alpha(t_r)) & y_\alpha(t_r)(\Sigma a_k \phi_k'(x_\alpha(t_r)))+x_\alpha(t_r) \end{pmatrix} \quad (2.11)$$

At $\lambda^*, e^*_*(t)$ has characterization according to Theorem 1.7 and hence $e^{*\alpha}_{\alpha *}(t)$ has either a straddle point extrema or at least $n + 1$ extrema in t $(r \geq n + 1)$.

2.2.2 CASE: BEST ROTATED MINIMAX APPROXIMATION ERROR FUNCTION HAS A STRADDLE POINT EXTREMA

Theorem 2.13: If $e^*_\alpha(t)$ has one straddle point and $r = n + 1$ extrema, then a necessary condition for L_n B.R.M.A. is that the derivative of the approximation in x_α is zero at the straddle point.

Proof: By Theorem 2.11, the rank of D must be less than $n + 1$. We assume the straddle point occurs at t_1 and t_2; i.e., $x_\alpha(t_1) = x_\alpha(t_2)$.

$$D = \begin{pmatrix} \phi_1(x_\alpha(t_1)) \ldots & \phi_n(x_\alpha(t_1)) & y_\alpha(t_1)(\Sigma a_k \phi_k'(x_\alpha(t_1)) + x_\alpha(t_1) \\ \phi_1(x_\alpha(t_1)) \ldots & \phi_n(x_\alpha(t_1)) & y_\alpha(t_2)(\Sigma a_k \phi_k'(x_\alpha(t_1)) + x_\alpha(t_1) \\ \phi_1(x_\alpha(t_3)) \ldots & \phi_n(x_\alpha(t_3)) & y_\alpha(t_3)(\Sigma a_k \phi_k'(x_\alpha(t_3)) + x_\alpha(t_3) \\ \vdots & & \\ \phi_1(x_\alpha(t_{n+1}) \ldots & \phi_n(x_\alpha(t_{n+1})) & y_\alpha(t_{n+1})(\Sigma a_k \phi_k'(x_\alpha(t_{n+1})) + x_\alpha(t_{n+1}) \end{pmatrix} \quad (2.12)$$

which implies

$$|D| = (y_\alpha(t_1) - y_\alpha(t_2))(\Sigma a_k \phi_k'(x_\alpha(t_1)) \begin{vmatrix} \phi_1(x_\alpha(t_1)) \dots & \phi_n(x_\alpha(t_1)) \\ \phi_1(x_\alpha(t_3)) \dots & \phi_n(x_\alpha(t_3)) \\ \phi_1(x_\alpha(t_{n+1})) \dots & \phi_n(x_\alpha(t_{n+1})) \end{vmatrix} . \tag{2.13}$$

The third factor is non-zero since the set of functions $\{\phi_k\}$ satisfies the Haar condition [10, p.91]. The first factor is never zero by the fact that the α-rotation of a continuous function is a Jordan arc. Hence the second term must be zero at a best rotated minimax approximation. This is the derivative of the approximation evaluated at the straddle point.

<u>Theorem 2.14</u>: If there exists $r = n + 1$ extrema then a necessary condition for L_n B.R.M.A. is the existence of two or more straddle points.

<u>Proof</u>: If $\exists$ one straddle point, then (2.13) is valid. If $\exists$ an additional straddle point, then the third factor of (2.13) must necessarily be zero.

2.2.3. POLYNOMIAL BEST NOTATED MINIMAX APPROXIMATION

Since the crucial property of the C-P matrix with respect to the Curtis and Powell theorems is its rank, then for polynomial approximation D can be written as

$$D = \begin{pmatrix} 1 & x_\alpha(t_1) \dots & x_\alpha^{n-1}(t_1) & y_\alpha(t_1)\,(\sum_{k=0}^{n-1} ka_k\, x_\alpha^{k-1}(t_1)) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 1 & x_\alpha(t_r) \dots & x_\alpha^{n-1}(t_r) & y_\alpha(t_r)\,(\sum_{k=0}^{n-1} ka_k\, x_\alpha^{k-1}(t_r)) \end{pmatrix} . \tag{2.14}$$

We note that if $\alpha \in R$ and we assume $r = n + 1$, then the matrix D of (2.14) is the numerator of the nth divided difference of the function $k_\alpha(t) = y_\alpha(t)(\sum_{k=0}^{n-1} ka_k\, x_\alpha^{k-1}(t))$ with respect to $x_\alpha(t)$.

If we assume the non-existence of a straddle point at α^*, then by Theorem 1.7b there exists $t_1, \dots t_{n+1} \in [0,1]$ such that $x_\alpha(t_1) < \dots < x_\alpha(t_{n+1})$ and

$$\pm (-1)^i \varepsilon = y_\alpha(t_i) - \sum_{k=0}^{n-1} a_k\, x_\alpha^k(t_i) \;,\; i = 1,\dots,n+1. \tag{2.15}$$

Hence we can solve for ε^*, a_k^* by the linear system

$$\begin{pmatrix} 1 & x_{\alpha^*}^{n-1}(t_1) & \dots & x_{\alpha^*}(t_1) & 1 \\ -1 & x_{\alpha^*}^{n-1}(t_2) & \dots & x_{\alpha^*}(t_2) & 1 \\ 1 & & & & \\ \cdot & & & & \\ \cdot & & & & \\ (-1)^n & x_{\alpha^*}^{n-1}(t_{n+1}) & \dots & x_{\alpha^*}(t_{n+1}) & 1 \end{pmatrix} \begin{pmatrix} \varepsilon^* \\ a_{n-1}^* \\ a_{n-2}^* \\ \cdot \\ \cdot \\ a_o^* \end{pmatrix} = \begin{pmatrix} y_{\alpha^*}(t_1) \\ y_{\alpha^*}(t_2) \\ \cdot \\ \cdot \\ \cdot \\ y_{\alpha^*}(t_{n+1}) \end{pmatrix} \tag{2.16}$$

<u>Proposition 2.15</u>: If the quadratic B.R.M.A. has no straddle points (and if $f^{(2)}(x) \underset{(<)}{>} 0$, $x \in (0,1)$, then $a_2^* \neq 0$.

<u>Proof</u>: By (2.16) we can solve for a_2^*; i.e.,

$$a_2^* = \begin{vmatrix} 1 & y_{\alpha^*}(t_1) & x_{\alpha^*}(t_1) & 1 \\ -1 & \cdot & \cdot & \cdot \\ & & \cdot & \\ 1 & \cdot & \cdot & \cdot \\ -1 & y_{\alpha^*}(t_4) & x_{\alpha^*}(t_4) & 1 \end{vmatrix} / \Delta \tag{2.17}$$

The denominator of (2.17), Δ , is not zero provided straddle point extrema do not exist, by existence and uniqueness of α^*-rotated minimax approximation. Using (1.1) the numerator of (2.17) can be reduced to

$$\begin{vmatrix} 1 & t_1 & f(t_1) & 1 \\ -1 & t_2 & f(t_2) & 1 \\ +1 & t_3 & f(t_3) & 1 \\ -1 & t_4 & f(t_4) & 1 \end{vmatrix} . \tag{2.18}$$

By [1, p.77, prob. 8] $k(t) = c_1 + c_2 t + c_3 f(t)$ for real c_i and $t \in [0,1]$, can have at most two zeroes in [0,1]. Hence the last three columns of (2.18) are linearly independent. The first column is independent of the last three columns since $k(t)$ can change sign at most two times on [0,1].

<u>Proposition 2.16</u>: If the quadratic B.R.M.A. has $a_2^* = 0$, $r = n + 1 = 4$, no straddle point extrema, and $\varepsilon^* \neq 0$, then $a_1^* = 0$.

<u>Proof</u>: We can write the determinant of the matrix (2.14) as

$$|D| = a_1^* \begin{vmatrix} 1 & x_{\alpha^*}(t_1) & x^2_{\alpha^*}(t_1) & y_{\alpha^*}(t_1) \\ \cdot & & & \\ \cdot & & & \\ 1 & x_{\alpha^*}(t_4) & x^2_{\alpha^*}(t_4) & y_{\alpha^*}(t_4) \end{vmatrix} + 2a_2^* \begin{vmatrix} 1 & x_{\alpha^*}(t_1) & x^2_{\alpha^*}(t_1) & y_{\alpha^*}(t_1) x_{\alpha^*}(t_1) \\ \cdot & & & \\ \cdot & & & \\ 1 & x_{\alpha^*}(t_4) & x^2_{\alpha^*}(t_4) & y_{\alpha^*}(t_4) x_{\alpha^*}(t_4) \end{vmatrix}$$

which, by our assumption and (2.16), reduces to

$$|D| = - a_1^* \varepsilon^* \Delta .$$

By hypothesis, ε^* and $\Delta \neq 0$. Hence, at a B.R.M.A., $a_1^* = 0$.

2.3 <u>THE NON-UNIQUENESS OF α^*</u>

We will prove by constructing an example that α^* need not be unique for P_o best rotated minimax approximation. We will need a result from [8, Thm. 6.6] which states that the error function of a best rotated constant minimax approximation must have three equioscillating extrema or a straddle point. Let f be the continuous function of Fig. 2.1;

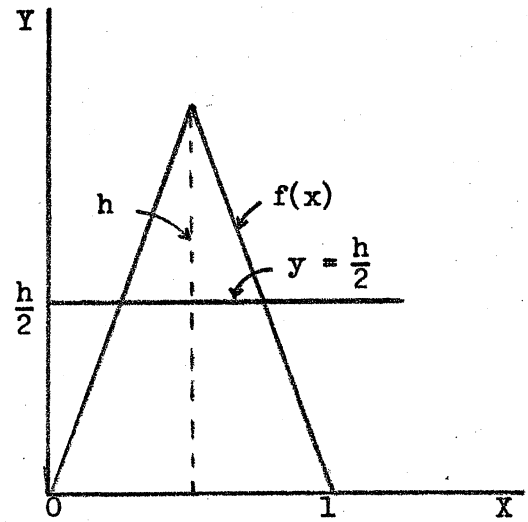

Fig. 2.1

i.e., an isoceles triangle without its base and defined on [0,1]. Let h be the height of the function. Evidently $y = \frac{h}{2}$ satisfies the necessary conditions for a B.R.M.A. Another candidate for α^* is the rotation of f sketched in Fig. 2.2, where the left leg of f has been rotated so that it is parallel to the X axis. A third possibility, a ninety degree rotation of f in Fig. 2.1, can be ruled out because the minimax error in Fig. 2.2 is smaller. If we ignore non-uniqueness due to symmetry, then, setting $h = \frac{\sqrt{3}}{2}$ the resulting minimax errors at $\alpha = 0$ and α of Fig. 2.2 are equal. Hence α^* is non-unique. We can also conclude that if $h < \frac{\sqrt{3}}{2}$ then $\alpha^* = 0 \in R$, and if $h > \frac{\sqrt{3}}{2}$ then $\alpha^* \notin R$.

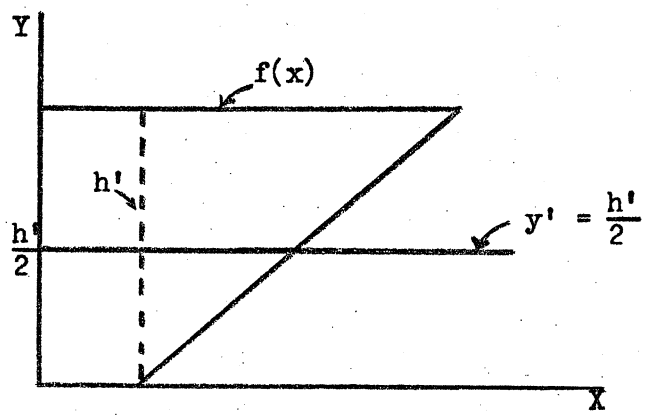

Fig. 2.2

3. BEST ROTATED MINIMAX APPROXIMATION: COMPUTATION

3.1 ALGORITHM FOR COMPUTING B.R.M.A.

In this chapter we shall be concerned with polynomial approximation and hence with the error function (1.4) which has n linear and one non-linear parameter. The technique we describe for computing α^* and the B.R.M.A. is an iterated linear programming approach due to Esch and Eastman [5] which has the advantage of not depending on characterization properties of equioscillating extrema for minimax approximation.

We wish to minimize h subject to the constraints

$$|h| \geq |e(t_j, \lambda)| \qquad j = 1,\ldots,N$$

which can be rewritten

$$h - e(t_j, \lambda) \geq 0 \qquad j = 1,\ldots,N$$

$$h + e(t_j, \lambda) \geq 0$$

and is equivalent to

$$\left.\begin{aligned} h - y_\alpha(t_j) + \sum_{k=0}^{n-1} a_k x_\alpha^k(t_j) &\geq 0 \\ h + y_\alpha(t_j) - \sum_{k=0}^{n-1} a_k x_\alpha^k(t_j) &\geq 0 \end{aligned}\right\} \quad j = 1,\ldots,N. \qquad (3.1)$$

Equation (3.1) is not in linear programming form since the parameter α enters non-linearly. Our goal is to linearize (3.1) in α so that we may use linear programming methods for its solution.

Let $\alpha = \alpha_o + \delta\alpha$. Then

$$y_\alpha(t) = -x_{\alpha_o}(t) \sin \delta\alpha + y_{\alpha_o}(t) \cos \delta\alpha$$

$$\doteq -\delta\alpha\, x_{\alpha_o}(t) + y_{\alpha_o}(t)$$

$$x_\alpha(t) = x_{\alpha_o}(t) \cos \delta\alpha + y_{\alpha_o}(t) \sin \delta\alpha \tag{3.2}$$

$$\doteq x_{\alpha_o}(t) + \delta\alpha\, y_{\alpha_o}(t)$$

$$x_\alpha^{n-1}(t) = (x_{\alpha_o}(t) \cos \delta\alpha + y_{\alpha_o}(t) \sin \delta\alpha)^{n-1}$$

$$x_\alpha^{n-1}(t) \doteq x_{\alpha_o}^{n-1}(t) + \delta\alpha\,(n-1)x_{\alpha_o}^{n-2}(t)y_{\alpha_o}(t).$$

Using (3.2) the system (3.1) can be linearized in the following way where for convenience we shall write only the second term in (3.1):

$$h + (1 - a_1\,\delta\,\alpha)y_{\alpha_o}(t_j) - a_o - (a_1 + \delta\,\alpha)x_{\alpha_o}(t_j)$$

$$- a_2 x^2_{\alpha_o}(t_j) - 2a_2\,\delta\,\alpha\, x_{\alpha_o}(t_j)y_{\alpha_o}(t_j)$$

$$-a_3 x^3_{\alpha_o}(t_j) - 3a_3\,\delta\,\alpha\, x^2_{\alpha_o}(t_j)y_{\alpha_o}(t_j) \qquad j = 1,\ldots,N \tag{3.3}$$

$$-\ldots - a_{n-1}x^{n-1}_{\alpha_o}(t_j) - (n-1)a_{n-1}\delta\,\alpha\, x^{n-2}_{\alpha_o}(t_j)y_{\alpha_o}(t_j) \geq 0.$$

Finally, we rewrite (3.3) as

$$y_{\alpha_o}(t_j) + h' - A_o - A_1 x_{\alpha_o}(t_j) - A_2 x^2_{\alpha_o}(t_j) - B_2 x_{\alpha_o}(t_j)y_{\alpha_o}(t_j)$$

$$-A_3 x^3_{\alpha_o}(t_j) - B_3 x^2_{\alpha_o}(t_j)y_{\alpha_o}(t_j) - \ldots \qquad j = 1,\ldots,N$$

$$-A_{n-1}x^{n-1}_{\alpha_o}(t_j) - B_{n-1}x^{n-2}_{\alpha_o}(t_j)y_{\alpha_o}(t_j) \geq 0. \tag{3.4}$$

On a given iteration, we start with a guess α_o and obtain as a solution of the linear program the values $A_o,\ldots,A_{n-1}$, $B_2,\ldots,B_{n-1}$ which gives the best approximation of (3.4) over the discrete point set $\{t_j\}$. We can calculate

$$\delta\,\alpha = B_2/2A_2$$

and replace α_o by the hopefully better estimate $\alpha_o + \delta\,\alpha$. In order to limit $\delta\,\alpha$ so as to prevent the process from moving too far from the region in which the linearization is accurate,

constraints of the form

$$
\begin{aligned}
&2\,\varepsilon\,|A_{2(old)}| \pm B_2 \geq 0 \\
&\quad\vdots \\
&(n-1)\,\varepsilon\,|A_{n-1(old)}| \pm B_{n-1} \geq 0
\end{aligned}
\tag{3.5}
$$

are incorporated into the program. The iterative process is halted when $\delta\,\alpha$ becomes less than some preassigned tolerance. At a solution, $A_i \doteq a_i$ and $h^* \doteq h'$.

3.2 COMPUTATIONAL EXPERIENCE

When α is given, $e^*(\alpha)$ is easily computed by standard linear programming techniques. Since the parameter space of α for the α-rotation minimax error, $e^*(\alpha)$ is $[0,\pi]$, a parameter search in α for the B.R.M.A. is feasible. With only a few exceptions (functions in Table II not appearing in Table I), the results of the following section were computed in this manner. Although a parameter search provides the only real assurance that a B.R.M.A. has been found, it is something less than a practical solution. The algorithm of Section 3.1 was implemented in the case of quadratic approximation. The linearization tolerance was set at .1, the iterative tolerance at 10^{-5}, α_o was taken at 0, and $A_{2(old)}$

at 1 for the initial iteration. The algorithm proved to be very efficient and accurate requiring no more than thirteen iterations, if it converged at all. Normally, for the last three iterations, the linear constraint was inactive. However, the algorithm did not converge for the functions: $x^2e^{-x^2}$, e^{-x^2}, x^2e^{-x}. For all three cases the error function of the B.R.M.A. has four equioscillating extrema or critical points at α^*. For all other cases, the B.R.M.A. error function had five critical points at α^*. It is also true that for the functions $x^2e^{-x^2}$, e^{-x^2}, x^2e^{-x}, $e^*(\alpha)$ has a very flat slope for a large neighborhood of α^*. At present the question is open whether the non-convergence of the algorithm is due to round-off error or to theoretical reasons associated with the fact that the B.R.M.A. error function has only the necessary number of critical points at α^*. A similar phenomenon was observed for the function e^{2x}. For the function e^{2x}, $e^*(\alpha)$ has two relative mins, one at $\alpha = .082$, with five critical points and another at $\alpha = 1.13$ with four critical points. Regardless of the starting value α_o or the linearization tolerances, the algorithm converged to $\alpha^* = .082$.

3.3 COMPUTED RESULTS

In this section we discuss some of the numerical results we have obtained for second and third degree polynomial approximation, using the algorithm of Section 3.1. and a parameter search program, for computing the optimal orientation α^*. These results are summarized in Tables I and II. For all functions listed, $x \in [0,1]$,

the discretized point set consists of 101 points evenly distributed over the interval. In Table I, for each function, the first entry under minimax error, for quadratic and cubic approximation, is the minimax error for $\alpha = 0$. The second row gives the minimax error at a relative min, the angle at which the relative min is achieved, and the number of critical points of the error function at the relative min. Often two relative mins were found. In these cases, the minimax error, the angle, and the number of critical points are again tabulated. In Table II, we compare the unrotated minimax quadratic error with the best rotated quadratic minimax error and the unrotated minimax cubic error. The minimax errors of the third and fourth column have the same number of effective parameters.

3.4 DISCUSSION

From Table II it is evident that the minimax error at a B.R.M.A. may or may not be smaller than the error at a cubic minimax approximation. For the examples given, the B.R.M.A. often has significantly smaller error than the unrotated minimax approximation of same degree. We note that the ratio of quadratic B.R.M.A. error to second degree minimax approximation is a factor of nineteen for the function e^{2x} and a factor of thirty-eight for the function e^{x^2}. Typical improvement seems to be in the order of factors of five to ten.

An examination of the data in Table I reveals that for most of the cases examined, the B.R.M.A. error function had n+2 equioscillating extrema, rather than the necessary n+1. In earlier stages of this study, it was anticipated that n+2 equioscillating extrema characterized, in some way, a B.R.M.A. However, this is not the case. The example $x^2e^{-x^2}$, quadratic approximation, demonstrates that n+2 equioscillating extrema is not necessary at a B.R.M.A. The function e^{2x}, cubic approximation, demonstrates that the condition is not sufficient. The function e^{2x}, quadratic approximation, had a relative min with n+1 equioscillating extrema, demonstrating that even for convex functions, n+2 equioscillating extrema is not necessary at a relative min. For the function $x^2e^{-x^2}$, cubic approximation, n+2 equioscillating extrema were observed at $\alpha = 1.306$. At this point, $e^*(\alpha)$ was not at a relative min. Hence n+2 equioscillating extrema is not locally sufficient.

The function $\sqrt{x}$, $x \in [0,1]$, has been useful as a counter-example to many conjectures concerning the characterization of α^*. It may be conjectured that the angle of rotation for which the range of the derivative in absolute value is a minimum should be α^*. However, this is not true for $\sqrt{x}$. It may also be conjectured that the angle for which the modulus of continuity is least should be α^*. This is also contradicted by the $\sqrt{x}$ function.

TABLE I

Function $x \varepsilon [0,1]$	Quadratic approximation minimax error	α^*	c.p.	Cubic approximation minimax error	α^*	c.p.
x^3	.03125			0		
	.00538	.2782	5			
x^4	.06345			.00781		
	.01114	.4130	5	.00169	.0647	6
				.00648	.8679	6
e^x	.000875			.000543		
	.000250	.2864	5	.0000146	.0817	6
				.0000251	.62	6
e^{2x}	.12237			.01501		
	.00645	.0818	5	.000871	.0229	6
	.03441	1.13	4	.00138	.1957	6
sin2x	.02192			.00418		
	.00823	-.1279	5	.00533	-.268	6
				.000321	.088	6
xe^{-x}	.00818			.000706		
	.000659	.6144	5	.0000414	.08	6
				.000187	.970	6
x^2e^{-x}	.01128			.00170		
	.01092	.25	4	.000974	-.588	6
e^{-x^2}	.01787			.000658		
	.01753	-.216	4	.000377	-.044	6
xe^{-x^2}	.00617			.00452		
	.00481	-.0750	5	.000882	.448	6
				.00285	-.433	6
$x^2e^{-x^2}$	.02607			.00201		
	.02405	.40	4	.00109	.34	5
$e^{x^2}/(x+1)$	.01280			.00664		
	.00584	.0873	5	.00584	.09	5
$e^x/(x+1)$	.00123			.000545		
	.000546	-.1484	5	.000537	-.076	5

TABLE II

Function $x \in [0,1]$	Quadratic minimax error	Quadratic B.R.M.A.error	Cubic minimax error	α^*
x^3	.03125	(.005378) .00538*	.0	.2782* (.41305)
x^4	.06345	.01114*	.00781	.4130*
x^5	.09216	.01647*	.01971	.4922*
x^6	.11704	.02121*	.03326	.5438*
e^x	.000875	.0002493 .000250*	.000543	.2865 .2864*
e^{2x}	.12237	.006454 .00645*	.01501	.08178 .0818*
e^{3x}	.74832	.05546 .05547*	.13494	.038806 .0288*
e^{4x}	3.32688	.30374 .30375*	.77822	.010848 .0108*
e^{-x}	.00322	.000111	.000198	-.6757
e^{-2x}	.01656	.00102	.00203	-.5445
e^{-3x}	.03726	.00319	.00672	-.5246
sin2x	.02192	.00823	.00418	-.1279
e^{x^2}	.05894	.00154	.01268	.2223
e^{-x^2}	.01787	.01753	.000658	-.216
xe^{-x}	.00818	.000659	.000706	.6144
xe^{-x^2}	.00617	.00481	.00452	-.0750
x^2e^{-x}	.01128	.01092	.00170	.25
$x^2e^{-x^2}$	.02607	.02405	.00201	.40
$e^{x^2}/(x+1)$	.01280	.00584	.00664	.0873
$e^x/(x+1)$	.00123	.000546	.000545	-.1484

*checked using double precision routines.

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