Expected Utility and the Michaud Efficient Frontier

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For most of the second half of the 20th century, modern neoclassical finance, and more generally much of social science, has been based on the von Neumann and Morgenstern (VM) (1944) game theory expected utility axioms as a prescription for rational decision making under uncertainty.[1] Traditional criticism of the Markowitz (1952, 1959) mean-variance (MV) efficient frontier holds that it is inconsistent with VM expected utility maximization except under the unrealistic conditions of an exact normal return distribution or quadratic expected utility function.

Markowitz addressed the issue of the consistency of the Markowitz efficient frontier with investor expected utility estimation in Levy and Markowitz (LM) (1979) and Kroll, Levy, and Markowitz (KLM) (1984). They argue that portfolios on the Markowitz efficient frontier provide useful and convenient approximations to portfolios that maximize investor expected utility. In the proposed two-step estimation process, compute the Markowitz MV efficient frontier for an investor's estimates of risk and return then find the portfolio on the Markowitz frontier that maximizes a MV approximation of the investor's expected utility function.[2] The Markowitz LM and KLM framework is proposed to provide convenient approximations to portfolios that maximize investor expected utility for portfolios on the Markowitz MV frontier.

It is however of interest to consider the current state of expected utility theory in the context of MV efficient frontiers. Allais (1953) and Kahneman and Tversky (KT) (1979) argue convincingly that human decision making under uncertainty is inconsistent with VM game theory utility axioms. Quiggin (1982, 1993) proposed the rank-dependent expected utility (RDEU) algorithm as a rational expected utility framework that resolves the issues of transitivity and stochastic dominance that arose in the KT framework. Quiggin observes: "The solution is to arrange the states of the world so that the outcomes they yield are ordered from worst to best, then to give each state a weight that depends on its ranking as well its probability."[3] The Quiggin RDEU expected utility theory is the basis of Tversky and Kahneman (1992) cumulative prospect theory, often considered the state-of-the-art of expected utility theory. RDEU theory is not consistent with Markowitz two-step expected utility approximation.

Markowitz optimization is an operations research algorithm that is insensitive to the statistical uncertainty in investment information. As a consequence the Markowitz procedure is highly unstable, the notion of optimality ambiguous, and resulting optimized portfolios often investor intuition unrealistic. Michaud (1998) and Michaud and Michaud (2008a,b) introduced the

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[1] E.g., Fennema and Bezembinder (1995).

[2] A linear mean-variance approximation to continuous functions in engineering and many scientific applications is a widely used tool in practice.

[3] Quiggin (1993, p. 63).

Michaud MV efficient frontier as a generalization of the Markowitz frontier based on statistical estimation principles to address the limitations of the Markowitz procedure in practice.[4] The Michaud MV optimization procedure reduces instability and ambiguity in portfolio optimality and solutions are often consistent with investor intuition. In rigorous simulation studies the Michaud algorithm has been shown to have superior out-of-sample performance relative to Markowitz on average (Michaud, 1998, Ch. 6; Markowitz and Usmen 2003). The Michaud procedure is fundamentally a statistical estimation procedure that is a significant departure from Markowitz that considers the estimation error uncertainty endemic in investor estimates of portfolio risk and return.

In order to describe a relationship between Michaud optimization and RDEU expected utility theory we need to further discuss some properties of the Michaud optimization algorithm. In the Michaud procedure there is a difference between the theoretical Michaud MV efficient frontier and the MV efficient frontier averaging algorithm used to compute it in practice. The expected return-rank algorithm averages the return ranks of the multivariate normal simulated Markowitz MV efficient frontiers to compute the Michaud frontier. In the procedure you have to decide how many rank portfolios to compute for each return-rank associated simulated MV efficient frontier. The limit of increasing the number of rank portfolios defines the continuous theoretical Michaud efficient frontier. Any point on the theoretical Michaud efficient frontier can be approximated with any desired accuracy simply by increasing the number of computed rank portfolios of simulated MV efficient frontiers. In practice, computing a hundred equal-return rank portfolios may be sufficient for many applications.[5] At the limit of increased ranks, the continuous Michaud frontier replaces the continuous Markowitz frontier as an improved definition of portfolio optimality in the MV optimization framework.

The necessary step in RDEU utility theory is to convert the probability distribution of outcomes into ranks. This is precisely the function of the Michaud expected return-rank computational algorithm. The procedure converts the assumed investor portfolio risk-return distribution into a return ranking of MV optimized portfolios. In applications, a probability transformation function can be imposed on the ranks expressing various investor utility characteristics. A linear or concave increasing function are examples that may express increasing risk aversion as a function of increasing ranks. Unlike Markowitz two-step estimation, the Michaud algorithm is no approximation but exactly consistent with the RDEU expected utility framework.

The problem that needs to be addressed is that the Michaud efficient frontier may not always have an increasing concave relationship of expected return relative to portfolio standard deviation or ranks in practice. This is the problem discussed in Michaud and Esch (2017). Michaud optimization is not an operations research algorithm. As multivariate statistical estimation, Michaud optimization is subject to the well-known limitations of badly defined statistical practice familiar in many multivariate regression studies and applications. In the case of portfolio

[4] Michaud (1989).

[5] Note that the return-rank algorithm is one of a number of algorithms that may be convenient for computing the Michaud frontier. One alternative is an arc-length averaging of simulated frontiers that can deal with issues associated with the shape of the frontier and hard to compute points. Interpolation methods can also be used to speed up applications for practice.

optimization, the inputs may poorly reflect an opportunity to diversify a portfolio when only two assets out of ten have positive estimates for describing the composition of optimized portfolios on the efficient frontier. Other examples include highly heterogeneous estimates where a small subset dominate diversification. A necessary condition for the investment value of any portfolio optimization is a set of assets and risk-return estimates that are considered to provide significant diversification benefit across the spectrum of portfolios on the efficient frontier. As in any statistical procedure Michaud optimization can be misused; the data has to make sense for the purpose of the application. Michaud optimization requires an appropriate set of risk-return estimates in order to facilitate construction of a well-defined investment program.[6]

In a well-defined investment program, Michaud optimization will generally provide monotone increasing expected return-ranked optimized portfolios. In this case, the simplest probability weighting function is no weighting at all reflecting investor increasing risk aversion. More generally, RDEU theory allows many alternative frameworks for expected utility estimation with probability transformation functions. The S-shaped function associated with KT can be used to overweight extreme low probability events and underweight extreme positive events. A natural investment strategy framework for the S-shaped function is a two-portfolio long-short investment strategy as described in Michaud (1993). While a concave transformation function expressing increasing risk aversion is typically associated with Markowitz frontiers, it may be of interest to consider the implications of a convex transformation function in the context of lotteries and gambles. Clearly there are many possible examples beyond these simple cases that could be of interest. It is interesting to speculate that many heretofore unexplored investment strategies may also be defined with alternative transformation functions all leading to expected utility consistent strategies.

It is important to note that there is an additional fundamental mismatch between the Quiggin RDEU framework and the estimate uncertainty conditional Michaud efficient frontier. This is the issue of objective probabilities (roulette lotteries) versus subjective probabilities (horse lotteries). To connect RDEU theory to probability estimate uncertainty implicit in Michaud MV optimization requires the Choquet (1953-54) integral. In this context probabilities need to be replaced with a σ-field of subsets of some space Ω where the subsets have the same behavior as utility functions on probabilities.[7] As Wakker (1990) notes, assuming stochastic dominance, RDEU theory with uncertain probabilities is the same as in the certainty case beyond interpretation. No fundamental change in application of RDEU theory to Michaud optimization is required.

Rationality properties should be a minimal condition for the definition of a valuable investment strategy. RDEU expected utility consistent MV optimization provides a rich convenient framework for reliable and investment meaningful asset management in practice that did not exist before. The investor convenient Markowitz optimization framework, in the context of the Michaud generalization, provides a solid theoretical basis that should reassure investors and theoreticians alike.

[6] One simple method is to run a Markowitz optimization that excludes many assets in the optimized portfolios.
[7] See the discussion in Quiggin (1993, sec. 5.7).

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